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THE

STUDENT'S ALGEBRA.

THE

STUDENT'S ALGEBRA.

THE S.A. 1829

STUDENT'S ALGEBRA,

WITH

NOTES AND OBSERVATIONS;

IN WHICH

THE ELEMENTS AND PRACTICE OF THAT SCIENCE ARE RENDERED FAMILIAR, AND PRACTICALLY ADAPTED TO THE USE OF SCHOOLS.

BY JOHN DARBY,

TEACHER OF THE MATHEMATICS, AND MASTER OF AN ACADEMY AT LEEDS, AUTHOR OF A KEY TO BLAND'S ALGEBRAICAL PROBLEMS, &c.

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1829.

78.

ENTERED AT STATIONERS' HALL.

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PREFACE.

ALGEBRA, the logic of Arithmetic, or universal reasoning, is a science which far surpasses Arithmetic and Geometry in point of application, these two being chiefly fundamental sciences, whilst Algebra not only investigates rules and theorems of its own, but, by an admirable analysis, extends to every branch of mathematical learning, solving with accuracy, brevity, and elegance, the most abstruse and difficult problems. It was by the aid of this science, combined with Geometry, that Newton settled the whole system of the visible world. Hence it is one of the most important studies, at once useful and elegant, which can be pursued during the period of education.

Of these truths, every one acquainted with Algebra must be convinced. It is the most refined species of logic; an expert Algebraist cannot fail to be an acute reasoner, not only in the affairs of common life, but in all pursuits of Literature and Philosophy. Style itself is characterised by an acquaintance with it. The certainty of its demonstrations allures the mind to a habit of clear and accurate reasoning, which is necessary, in order to keep the imagination in due subjection to the judgment, and prevents the understanding from becoming the dupe of our own weakness, or from being led captive by the wit and eloquence of others.

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PREFACE.

ALGEBRA is the logic of Arithmetic. It is an abstraction of those operations, of which Arithmetic is an exemplification of particulars. He who understands Algebra must understand Arithmetic, of which science it illustrates the reason of every operation. Hence, Algebra is one of the most important studies, at once useful and elegant, which can be pursued during the period of education.

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If so important an art has been less taught than it deserves to be in the ordinary Education of British youth, the cause appears to be the want of a neat and compact Elementary Treatise, adapted to the practical use of the schoolmaster, and also, to that economy of price which schoolmasters are obliged to respect, for the satisfaction of parents.

and compact Elementary Treatise, adapted to the practical use of the school-master, and also, to that economy of price which school-masters are obliged to respect, for the satisfaction of parents.

Both these purposes will, it is presumed, be found to be effected by the present work; the Author is himself a school-master, and, a teacher of the mathematics of nearly twenty years experience; and having in his profession had the good fortune to acquire a fair reputation as a teacher, in one of the most populous towns in England, he may be presumed to be qualified for the task which he has undertaken. None but school-masters can estimate a school-master's difficulties, and few can have a chance of removing them but those who have felt them. It is this circumstance which emboldens the Author of this treatise to calculate on its success. He affects to have made no discoveries, but cherishes the useful ambition of rendering easy and practicable the acquirement of a useful and necessary branch of knowledge.

It need scarcely be remarked to intelligent teachers, that, before a youth commences this study, he ought to be moderately acquainted with the Elements of Arithmetic, and especially versed in the working of Fractions. These facilities may be acquired by any system of Arithmetic which may be agreeable to the master, for Algebra is a science which grows out of the general principles of Arithmetic, and its study may follow any method by which those principles are taught.

Classical, Commercial, and Mathematical
Academy, Trafalgar-Street, Leeds.

June, 18, 1829.

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Classical, Commercial, and Mathematical Academy,
Trafalgar-street, Leeds, January 21st, 1829.

DEFINITIONS.

1. The Sign + (*plus*) signifies that the quantity to which it is prefixed, is to be added. Thus $a + x$, (pronounced a plus x) denotes that x, or the number or quantity which it represents, is to be added to a.

2. The Sign — (*minus*) signifies that the quantity to which it is prefixed, is to be subtracted. Thus $a - x$ (pronounced a minus x) denotes that x, or the number which it represents, is to be subtracted from a.

N. B. The quantities to which the Sign + is prefixed are called positive, and those to which the Sign — is prefixed are called negative quantities. But a quantity without a Sign is always positive.

3. The Sign × (*multiplication*) signifies that the quantities between which it stands, are to be multiplied together. Thus $a \times x$, (pronounced a multiplied by x, or a into x) makes ax. The product of single letters is expressed by writing them down one after the other. Thus, the product of a, b, and c, is abc, or it may be, bca, or cba, or acb, or bac, or cab, the product not being affected by taking any of the letters first;—see the following example in figures:—

Let it be required to find the product of 4 and 6?

4	or	6
6		4
24		24

Here the product is the same when 4 is made the multiplicand and 6 the multiplier, as when 6 is made the multiplicand and 4 the multiplier.

4. The Sign . (*a period*) is frequently used for $\times$, thus $a \,.\, x$ denotes that a is to be multiplied into x, and makes ax, as in definition 3. Also, $(a+x) \,.\, (a+x)$, or which is the same thing $(a+x) \times (a+x)$, denotes that $a+x$ is to be multiplied by $a+x$.

5. The Sign $\div$ (*division*) when placed between two quantities denotes, that the first quantity is to be divided by the second. Thus $a \div x$ signifies that a is to be divided by x, or $\frac{a}{x}$ has the same signification.

Also, $(a+x) \div (a-x)$, or $\frac{a+x}{a-x}$ signifies that $a+x$ is to be divided by $a-x$.

6. The Sign : *as*, or *to*, and :: *so is*, signifies an equality of ratio. Thus $a : b :: c : d$ denotes that a has the same ratio to b that c has to d; and read or pronounced, as a is to b so is c to d.

7. The Sign $=$ (equality) when placed between two quantities, signifies that they are equal in value. Thus $4+5=9$, denotes that 4 added to 5 is equal to 9; also, $4-1=3$, denotes that 1 being subtracted from 4, the remainder is equal to 3.

8. The powers of Algebraic quantities are expressed by placing a small figure over the quantity, equivalent to the number of factors, which is called the index of the power.

Thus, $x \times x$ signifies the square of x, and is expressed by ..... x^2
——— $x \times x \times x$ .... the cube of x, ...by x^3
——— $x \times x \times x \times x$ .. the fourth power of x, by x^4
——— x^m denotes x in the m^{th} power, &c. &c.

Also, $(a+x) \times (a+x)$ signifies the square of $a+x$, and is expressed by $(a+x)^2$, or $\overline{a+x}|^2$; $(x+y) \cdot (x+y) \cdot (x+y)$ signifies the cube of $x+y$, and is ex-

pressed thus $(x+y)^3$, or $\overline{x+y}|^3$; and so on for the 4th, 5th, or any other power. These small figures are called the indices.

N. B. A letter or any quantity without a small figure placed over the right hand corner, is supposed to have 1 for an index. Thus x is the same as x^1.

9. The Sign $\sqrt{\quad}$ denotes the square root of the quantity to which it is prefixed. Thus, $\sqrt{25}$, signifies that the square root of 25 is to be extracted, or $\frac{1}{2}$ being placed over the right hand corner has the same effect. Thus, $\overline{25}|^{\frac{1}{2}}$ signifies that the square root of 25 is to be taken.

Therefore $\sqrt{\quad}$ or $\frac{1}{2}$ signifies the square root.
——— $\sqrt[3]{\quad}$ or $\frac{1}{3}$ ..... the cube root.
——— $\sqrt[4]{\quad}$ or $\frac{1}{4}$ ..... the 4th root.
——— $\sqrt[5]{\quad}$ or $\frac{1}{5}$ ..... the 5th root.
——— $\sqrt[n]{\quad}$ or $\frac{1}{n}$ ..... the n^{th} root, &c.

10. The fraction $\frac{2}{3}$ prefixed to a quantity, as $a^{\frac{2}{3}}$, signifies that the square of the cube root of a is to be taken, or $8^{\frac{2}{3}}$ signifies that the cube root of 8 is to be extracted, which is 2, and then the 2 to be squared, which is 4; therefore, $8^{\frac{2}{3}}$ is equal to 4.

Also, $8^{\frac{5}{3}}$ denotes the 5th power of the cube root of 8.

Also, $\overline{16}|^{\frac{5}{4}}$ denotes the 5th power of the 4th root of 16, &c.

Or thus,

$a^{\frac{2}{3}}$ signifies that a, or the quantity which it denotes, is to be squared, and the cube root of that power to

be extracted; also, $8^{\frac{2}{3}}$ denotes that 8 is to be squared, which is 64, then the cube root of this 64 to be extracted, which is 4, &c.

11. The coefficient of a quantity is the number or letter prefixed to it, and signifies how often the other is to be taken. Thus, $5x^2$ signifies that the x^2 is to be taken five times; here 5 is the coefficient to the x^2, and the small ^{2} over the x is called the index, and is read, five times x squared. Also, $4\sqrt{x}$, or $4x^{\frac{1}{2}}$, 4 being the coefficient, and denotes four times the square root of x.

N. B. When any quantity stands without a coefficient as x, it is supposed to have 1 for a coefficient, for x being the same as $1x$.

12. Quantities are said to be *like* which differ only in their coefficients. Thus, $4x$ and $-2x$ are like quantities, $3a^2$ and $4a^2$ are *like* quantities. But $2a$ and $3x$ are *unlike* quantities, a and x being different.

13. An irrational or surd quantity, is that which has no exact root, and can only be expressed by a radical sign or fractional index; as $\sqrt{3}$, or $3^{\frac{1}{2}}$; or $\sqrt[3]{x}$, or $x^{\frac{1}{3}}$, &c.

14. Simple quantities are those which consist of one term only, as $4x$, $5y$, $4xy$, &c.; and compound quantities, are those which consist of several terms, as $a+x$, $3x-4y$, $x^2+2xy+y^2$, &c.

15. A vinculum is a parenthesis ( ), or bar ——, used to collect several quantities into one; as $(x+y) \times a$, or $\overline{x+y} \times a$, signifies that the compound quantity $x+y$ is to be multiplied by the simple quantity a.

16. The *reciprocal* of any quantity or number, is that quantity or number inverted; as the reciprocal of x is $\frac{1}{x}$; the reciprocal of $\frac{a}{x}$ is $\frac{x}{a}$; the reciprocal of 5 is $\frac{1}{5}$ or .2, and so on.

17. The Sign $\therefore$ stands for *therefore*.

ADDITION.

ADDITION of Algebra is performed by connecting the quantities together by their proper signs, and may be divided into three cases.

CASE I.

RULE.—*When the quantities are like (Def. 12.) with the same sign, viz. all + or all —, add the coefficients into one sum, to which adjoin the letters common to each term, and prefix the sign.*

EXAMPLES.

$+ 6a$	$4x$	$- 3xy$	$2xy - 3x + 4a$
$+ 4a$	$3x$	$- 4xy$	$2xy - 4x + 6a$
$+ 3a$	$5x$	$- 7xy$	$7xy - x + 4a$
$+ 2a$	$7x$	$- 6xy$	$9xy - 7x + 9a$
$+ 15a$*	$19x$	$-20xy$	$20xy - 15x + 23a$

* The reason of this rule is obvious, for $2a$ added to $3a$, make $5a$; and $5a$ added to $4a$, make $9a$, and $9a$ added to $6a$, make $15a$.

It is usual in Addition to begin at the bottom of the left hand column, as in the 4th question, begin with $9xy$.

EXAMPLES.

$-2xy$	$x^{\frac{1}{2}}+xy$
$3ax$	$3a+xy^2$
$+2xy^2$	$xy-3xy^2$
xy	$12-4a$
$3ax$	$2xy^2+4\sqrt{x}$
$-4ax$*	$-2xy^2-4$
$2ax-xy+2xy^2$	$-2xy^2+8+2xy-a+5x^{\frac{1}{2}}$

x^2y	$x^2+2xy+y^2$	$6-\sqrt{ax}+x^2$
$-xy^2$	$a-3ax-x^{\frac{1}{2}}$	x^2+xy-a
$3xy^2$	$\sqrt{x}+2ax-x^2$	$3x^2-4xy+2a$
$2x^2y$	$3x^2-2xy+2x^{\frac{1}{2}}$	$-4x+2\sqrt{ax}-a^{\frac{1}{2}}$
$-2xy^2$	$4y^{\frac{1}{2}}+6ax-32$	$\sqrt{a}-3\sqrt{x}+12$

$4x\ y$	$3x^2-2xy+5z$	$4a+bx-c$
$3a\ x$	$2x+ax+3b$	$3x-4y+z$
$-2x^2y$	$4x^3-3xz-4$	$4c+xy-z$
$5x\ y^2$	$7x^{\frac{1}{2}}+2ay-5a$	$-bx+xz+4$
$5a\ b$	$x^2+2xy-5z$	$3y+4x-y^{\frac{1}{2}}$

* In the first example, $-4ax$, $3ax$, and $3ax$, are like quantities, which by case second make $2ax$. Also, xy and $-2xy$, are like quantities, which by case second make $-xy$; and the remaining quantity $2xy^2$, must be placed in the line with its proper sign, as above.

SUBTRACTION.

RULE.—*Change the signs of the quantities to be Subtracted, or conceive them to be changed; and collect them together as in Addition.*

EXAMPLES.

$4x$	$3a-2x$	$+x$	$3x^2+2y+\sqrt{x}$
$2x$	$2a+3x$	$-x$	$-2x+\sqrt{x}-6$
$2x$	$a-5x$	$2x$	$3x^2+2y+2x+6$

$4x^2-3a$	$2x^2+4xy+2y^2$	$2xy+3a$
x^2+2a	$x^2+2xy+y^2$	$-2a+xy$

$3\sqrt{x+y}+4y$	$2\sqrt{x-y}+3y^3$	$x^3+2ax+y$
$-2\sqrt{x+y}-2y$	$2\sqrt{x+y}-y^3$	$x^2+2ax-a$

* In example second, $2a$, which is to be subtracted from $3a$, is conceived or called $-2a$, then by case second in Addition, their difference will be a. Also, $+3x$, which is to be subtracted from $-2x$, is conceived or called $-3x$, then by the same case their difference is $-5x$.

1. From x^2-xy take $xy-y^2$.

Ans. $x^2-2xy+y^2$.

2. From $\frac{1}{2}a+xy$ take $\frac{2}{3}a-xy$.

Ans. $\frac{1}{1\frac{1}{2}}a+2xy$.

3. From $5a$ take $2x+1$.

Ans. $5a-2x-1$.

4. From $25-3x$ take one quarter of that quantity.

Ans. $\frac{75-9x}{4}$

MULTIPLICATION.

MULTIPLICATION of Algebra, as in common Arithmetic, is finding the product of two or more quantities, and is commonly divided into three cases.

CASE I.

When the quantities are both simple.

RULE.—*Multiply the coefficients of the quantities together, and annex all the letters in both quantities, or their powers: prefixing the sign + or — to the product, according as the quantities have like, or unlike signs.*

In Multiplication, *like* signs produce +, and unlike signs —, that is, if the multiplicand and multiplier be both +, or both —, the product will be +; but if one be +, and the other —, the product will be minus, see notes 3rd and 4th.

EXAMPLES.

$2a$	$3ax$	$3a$	-2	$-3x$
$3x$	$2y$	$4a$	$+4$	$-2a$
$6ax$	$6axy$	$12a^2$	-8	$+6ax$
$3x$	$4x^3z$	$3x^{\frac{1}{3}}$	$2x^{\frac{2}{3}}$	$2x^{n}$
$-4y$	$2x^2y$	$-5x^{\frac{2}{3}}$	$-x^{\frac{3}{4}}$	x
$-12xy$	$8x^5yz$	$-15x$	$-2x^{\frac{17}{12}}$	$2x^{n+1}$

1. In the 1st example both the quantities are +, therefore, the product is plus, and is performed as follows, viz. $2\times3=6$, and $a\times x=ax$, hence ax being annexed to the 6 makes the product $6ax$.

2. In the 3rd example, $3\times4=12$, and $a\times a=a^{1+1}=a^2$; therefore the product is $12a^2$; a^1 being the same as a, every letter, without an index, is supposed to have 1 for an index, hence, multiplying the same letters or their powers together is nothing more than adding their indices together.

Thus, $a\times a=a^1\times a^1=a^{1+1}=a^2$; $x^2\times x=x^{2+1}=x^3$; $x^3\times x^2=x^{3+2}=x^5$; $x^{\frac{2}{3}}\times x^{\frac{1}{4}}=x^{\frac{2}{3}+\frac{1}{4}}=x^{\frac{8}{12}+\frac{3}{12}}=x^{\frac{11}{12}}$, &c.

3. When minus and plus quantities are to be multiplied together, as in the 4th example, viz. $-2\times+4$ makes -8, it being the same thing, as adding four times -2 together, which will make -8.

4. In the 5th example both the quantities are minus; then since $3x-3x=0$, $\therefore (3x-3x)\times-2a=-6ax+6ax=0$, because 0, multiplied by any quantity, produces nothing; and since the 1st term of the product, viz. $3x\times-2a$, (by note 3) makes $-6ax$; therefore the 2nd product, viz. $-3x\times-2a$ must make $6ax$, otherwise the whole product $-6ax+6ax$ cannot be equal to nothing.

1. Multiply $3ax$ by $2ay$.

2. Multiply $3xyz$ by abc.

3. Multiply $4x^{\frac{3}{5}}$ by $x^{\frac{7}{5}}$

4. Multiply $3x^{\frac{1}{2}}y$ by $-3xy^{\frac{1}{3}}$.

5. Multiply $-x^{n}$ by $-x^{\frac{1}{n}}$.

CASE II.

When one of the quantities is a compound, and the other a simple quantity.

RULE.—*Multiply each term of the compound quantity by the simple quantity, beginning at the left hand or first term of the compound quantity, and proceeding to the right hand or last term; placing the quantities or products one after another with their proper signs.*

EXAMPLES.

$2x + 3a$	$4a - x^2$	$x^2 - 2xy + y^2$
$2x$	$3ax$	$2xy$
$4x^2 + 6ax$	$12a^2x - 3ax^3$	$2x^3y - 4x^2y^2 + 2xy^3$

$3x^{\frac{1}{2}} + 3a + 2xy$	$2x - 2xy + y^{\frac{2}{3}}$
$4x^2$	$-3y^{\frac{1}{3}}$
$12x^{\frac{5}{2}} + 12ax^2 + 8x^3y$	$-6xy^{\frac{1}{3}} + 6xy^{\frac{4}{3}} - 3y$

$2x+y$	$5x^{\frac{1}{2}}+2xy+y^2$	$-x^2+2xy+y^2$
$3x$	$2x^{\frac{1}{2}}$	$-x$

$4x^{\frac{2}{3}}-yz-abx$	$x^3-3x^2y+3xy^2-y^3$
$2x^{\frac{3}{4}}$	$x^{\frac{1}{2}}$

CASE III.

When both the factors are compound quantities.

RULE.—*Multiply each term of the multiplier, into every term of the multiplicand, beginning at the left hand and proceeding to the right, as in case 2nd. setting down these products one after another with their proper signs; then add them together by the different cases in Addition.*

EXAMPLES.

$2x + 3y$	$x + y$
$3x + 2y$	$x + y$
$6x^2 + 9xy$	x^2+xy
$\quad 4xy+6y^2$	$\quad xy+y^2$
$6x^2+13xy+6y^2$	$x^2+2xy+y^2$

Surd quantities, under the same radical sign, are multiplied as rational quantities; only the product must be placed under the same radical sign. Thus $\sqrt{9}\times\sqrt{7}=\sqrt{9\times7}=\sqrt{63}$, $\sqrt{a}\times\sqrt{x}=\sqrt{ax}$, $\sqrt{4a^2x^2}\times\sqrt{6ax}=\sqrt{24a^3x^3}$, &c. see example fourth which is solved.

$x^2-2xy+y^2$ | $4\sqrt{a}+\sqrt{xy}$
$x-y$ | $2\sqrt{x}-\sqrt{z}$

$x^3-2x^2y+xy^2$ | $8\sqrt{ax}+2\sqrt{x^2y}$
$-x^2y+2xy^2-y^3$ | $-4\sqrt{az}-\sqrt{xyz}$

$x^3-3x^2y+3xy^2-y^3$ | $8\sqrt{ax}+2\sqrt{x^2y}-4\sqrt{az}-\sqrt{xyz}$

1. Multiply $x-y$ by $x+y$. Ans. x^2-y^2
2. Multiply $x^2+2xy+y^2$ by $x+y$.
Ans. $x^3+3x^2y+3xy^2+y^3$
3. Multiply $x^{\frac{2}{3}}+xy^{\frac{1}{3}}$ by $x-y$.
Ans. $x^{\frac{5}{3}}+x^2y^{\frac{1}{3}}-x^{\frac{2}{3}}y-xy^{\frac{4}{3}}$
4. Multiply $\sqrt{xy}-\sqrt{az}$ by $\sqrt{ax}+\sqrt{by}$.
Ans. $\sqrt{ax^2y}-\sqrt{a^2xz}+\sqrt{bxy^2}-\sqrt{abyz}$.
5. Multiply $\frac{1}{x}+\frac{1}{y}$ by $\frac{1}{x}-\frac{1}{y}$. Ans. $\frac{1}{x^2}-\frac{1}{y^2}$
6. Multiply x^m+y^n by $x+y$.

Ans. $x^{m+1}+xy^n+x^m y+y^{n+1}$

DIVISION.

Division of Algebraic quantities, as in common Arithmetic, is the reverse of Multiplication, and is commonly divided into three cases.

CASE I.

When the quantities are both simple.

RULE.—*Divide the coefficient of the dividend, by the coefficient of the divisor, to obtain the coefficient of the quotient; expunge the letters which are common to both quantities, and place the remaining ones to the coefficient of the quotient.*

If the signs of the dividend and divisor be *like*, the sign of the quotient will be +; but if the signs be *unlike*, the sign of the quotient will be minus.

In simple quantities, place the dividend above a small line, and the divisor under it, in the manner of a vulgar fraction.

EXAMPLES.

1. Divide $4x^2$ by $2x$.

$$2x)4x^2 \quad \text{or} \quad \frac{4x^2}{2x} = 2x.$$
$$2x$$

2. Divide $12axy$ by $4x$.

$$4x)12axy \quad \text{or} \quad \frac{12axy}{4x} = 3ay.$$
$$3ay$$

3. Divide x by $-x$.

$$\frac{x}{-x} = -1.$$

4. Divide $\frac{2a^2x^2}{5ax}$ by $\frac{4x}{ax}$

Here $\frac{2a^2x^2}{5ax} \div \frac{4x}{ax} = \frac{2a^2x^2}{5ax} \times \frac{ax}{4x} = \frac{2a^3x^3}{20ax^2} = \frac{a^2x}{10}$

5. Divide $-\frac{2x}{3}$ by $-\frac{4x^2}{5a}$

Here $-\frac{2x}{3} \times -\frac{5a}{4x^2} = \frac{10ax}{12x^2} = \frac{5a}{6x}$ (as in fractions).

EXAMPLES FOR PRACTICE.

1. Divide $12ax$ by $4a$. Ans. $3x$.

2. Divide $15xyz$ by $3xyz^{\frac{1}{2}}$. Ans. $5z^{\frac{1}{2}}$.

3. Divide $-27a^2x^3$ by $9ax^2$. Ans. $-3ax$.

4. Divide $56ax$ by $-8a^{\frac{1}{2}}x^{\frac{1}{2}}$. Ans. $-7a^{\frac{1}{2}}x^{\frac{1}{2}}$.

5. Divide $-\frac{2a}{4x}$ by $-\frac{3x}{4a}$ Ans. $\frac{2a^2}{3x^2}$

CASE II.

When the dividend is a *compound* quantity, and the divisor a *simple* one.

RULE.—*Divide each term of the dividend by the divisor separately, and the resulting quantities, with their proper signs, will be the quotient required.*

EXAMPLES.

1. Divide $2xy+4x^2$ by $2x$.

$$2x)2xy+4x^2$$

$$y+2x$$

2. Divide $8x^2y-12xy+4x^2z$ by $4xy$.

$$4xy)8x^2y-12xy+4x^2z$$

$$2x \quad -3 \quad + \frac{xz}{y}$$

Such terms in the dividend, of which the divisor is not a common multiple, may be placed as a fraction, see example second.

If the dividend be a *simple* quantity, and the divisor a *compound* one, they may be placed as a vulgar fraction, see example fourth.

3. Divide $6x^2y^2-4x^3y+8xy$ by $-2xy$.

$$-2xy)\,6x^2y^2-4x^3y+8xy$$
$$-3xy+2x^2-4$$

4. Divide x by $a+z$.

Here (by note) $\frac{x}{a+z}$ the quotient.

EXAMPLES FOR PRACTICE.

1. Divide $4x^2+6xy$ by $2x$. Ans. $2x+3y$.

2. Divide $12xy^2-6x^2y^2+30xyz$ by $-6xy$.
Ans. $-2y+xy-5z$.

3. Divide $-6ax-3a^2x^2-15a^3x^3$ by $-3ax$.
Ans. $2+ax+5a^2x^2$.

4. Divide $\frac{x-b}{8cd}$ by $\frac{3cx}{4d}$ Ans. $\frac{x-b}{6c^2x}$

5. Divide $9xy-12ax+14x^2y^2$ by $4xy$.
Ans. $\frac{9}{4}-3\frac{a}{y}+\frac{14}{4}xy$.

CASE III.

When the divisor and dividend are both *compound* quantities.

RULE.—*Arrange both the divisor and dividend according to the powers of the same letters, beginning with the highest power; then find how often the first term of the divisor, is contained in the first term of the dividend, and place the result for a quotient, observing that if the first term in the divisor, and the first term in the dividend, have like signs, the term in the quotient will be plus, if unlike signs, minus.*

Multiply each term in the divisor by this quantity, and place the product under the corresponding terms in the dividend; then proceed as in subtraction. To the remainder, bring down the next term, or as many terms as may be wanted.

If there be a remainder after the division, it must be placed in the quotient, in the form of a vulgar fraction, with its sign placed before it, as in examples 4th and 5th.

EXAMPLES.

1. Divide $x^3—3x^2y+3xy^2—y^3$ by $x—y$.

$$
\begin{array}{l}
x—y)x^3—3x^2y+3xy^2—y^3(x^2—2xy+y^2 \\
\quad\quad x^3— x^2y \\
\quad\quad\quad\quad —2x^2y+3xy^2 \\
\quad\quad\quad\quad —2x^2y+2xy^2 \\
\quad\quad\quad\quad\quad\quad\quad xy^2—y^3 \\
\quad\quad\quad\quad\quad\quad\quad xy^2—y^3
\end{array}
$$

In this example the terms are arranged according to the rule, beginning with x^3.

1. The first term, x, of the divisor, and the first term, x^3, of the dividend, have like signs, and x is contained in x^3, x^2 times; therefore, x^2 will be the first term in the quotient.

2. Multiply $x—y$, by x^2, and it gives $x^3—x^2y$.

3. Subtract $x^3—x^2y$, from $x^3—3x^2y$, and there remains $—2x^2y$; to this remainder, bring down the next term $+3xy^2$.

4. Here the first term of the dividend, $—2x^2y$, is a minus quantity, and the first term, x, of the divisor, is plus, that is, they have *unlike* signs, and x being contained in $—2x^2y$, $—2xy$ times; therefore, the term in the quotient will be $—2xy$.

5. Multiply and subtract as before, and the remainder will be xy^2; to this remainder bring down the next or last term y^3.

6. x being contained in xy^2, $+y^2$ times, put this term in the quotient and proceed as above; then the quotient will be found to be $x^2—2xy+y^2$.

2. Divide $x^2+2xy+y^2$ by $x+y$.

$$\begin{array}{l} x+y)x^2+2xy+y^2(x+y \\ \quad x^2+\ xy \\ \hline \qquad xy+y^2 \\ \qquad xy+y^2 \\ \hline \end{array}$$

3. Divide $6x^2+13xy+6y^2$ by $2x+3y$.

$$\begin{array}{l} 2x+3y)6x^2+13xy+6y^2(3x+2y \\ \quad 6x^2+\ 9xy \\ \hline \qquad 4xy+6y^2 \\ \qquad 4xy+6y^2 \\ \hline \end{array}$$

4. Divide $x^3-9x^2+27x-30$ by $x-3$.

$$\begin{array}{l} x-3)x^3-9x^2+27x-30(x^2-6x+9- \dfrac{3}{x-3} \\ \quad x^3-3x^2 \\ \hline \qquad -6x^2+27x \\ \qquad -6x^2+18x \\ \hline \qquad\qquad 9x-30 \\ \qquad\qquad 9x-27 \\ \hline \qquad\qquad\quad -\ 3 \\ \hline \end{array}$$

5. Divide x^2+1 by $x-1$.

$$\begin{array}{l} x-1)x^2+1(x+1+\dfrac{2}{x-1} \\ \quad x^2-x \\ \hline \qquad x+1 \\ \qquad x-1 \\ \hline \qquad\quad 2 \end{array}$$

In the fourth question, -3 remains, hence, according to note at end of the rule, it is placed in the quotient in the form of a fraction, with its proper sign.

6. Divide x by $x+y$.

$$x+y)x\ (1-\frac{y}{x}+\frac{y^2}{x^2}-\frac{y^3}{x^3}\ \&c.$$

$$\begin{array}{l}
x+y \\
\hline
-y \\
-y-\frac{y^2}{x} \\
\hline
\frac{y^2}{x} \\
\frac{y^2}{x}+\frac{y^3}{x^2} \\
\hline
-\frac{y^3}{x^2} \\
-\frac{y^3}{x^2}-\frac{y^4}{x^3} \\
\hline
\frac{y^4}{x^3}
\end{array}$$

7. Divide $x^3+x^{\frac{3}{2}}y^{\frac{3}{2}}+y^3$ by $x^{\frac{3}{2}}+x^{\frac{3}{4}}y^{\frac{3}{4}}+y^{\frac{3}{2}}$.

$$x^{\frac{3}{2}}+x^{\frac{3}{4}}y^{\frac{3}{4}}+y^{\frac{3}{2}})x^3+x^{\frac{3}{2}}y^{\frac{3}{2}}+y^3(x^{\frac{3}{2}}-x^{\frac{3}{4}}y^{\frac{3}{4}}+y^{\frac{3}{2}}.$$

$$\begin{array}{l}
x^3+x^{\frac{9}{4}}y^{\frac{3}{4}}+x^{\frac{3}{2}}y^{\frac{3}{2}} \\
\hline
-x^{\frac{9}{4}}y^{\frac{3}{4}}+y^3 \\
-x^{\frac{9}{4}}y^{\frac{3}{4}}-x^{\frac{3}{2}}y^{\frac{3}{2}}-x^{\frac{3}{4}}y^{\frac{9}{4}} \\
\hline
+x^{\frac{3}{2}}y^{\frac{3}{2}}+x^{\frac{3}{4}}y^{\frac{9}{4}}+y^3 \\
+x^{\frac{3}{2}}y^{\frac{3}{2}}+x^{\frac{3}{4}}y^{\frac{9}{4}}+y^3 \\
\hline
\end{array}$$

EXAMPLES FOR PRACTICE.

1. Divlde $x^2-2xy+y^2$ by $x-y$. Ans. $x-y$.
2. Divide $x^3+3x^2y+3xy^2+y^3$ by $x+y$.
Ans. $x^2+2xy+y^2$.
3. Divide $6x^4-96$ by $3x-6$.
Ans. $2x^3+4x^2+8x+16$.
4. Divide 1 by $1-a$. Ans. $1+a+a^2+a^3+a^4$, &c.
5. Divide $\frac{x+a}{2x-2b}$ by $\frac{x+b}{5x+a}$ Ans. $\frac{5x^2+6ax+a^2}{2x^2-2b^2}$

INVOLUTION.

INVOLUTION is the raising of powers, or the method of finding the square, cube, biquadrate, fifth power, &c. of any given quantity.

RULE.—*Multiply the given quantity into itself, as many times as there are units in the power, less one, and the last product will be the power required.*

If the sign of the quantity to be raised be +, all its powers will be +; if —, the powers will be + and — alternately.

The square of x, is $+x\times+x=x^2$; the square of $2x^2$ is $+2x^2\times+2x^2$ or $4x^4$; the square of $-2x$ is $-2x\times-2x=4x^2$; (by note third to Case I. in multiplication) the cube of $-2x$ is $-2x\times-2x\times-2x=4x^2\times-2x$ or $-8x^3$.

EXAMPLES.

Suppose x a quantity or root to be involved.

Then x^2 will be its sqnare,
x^3 ..... cube,
x^4 ..... 4th power,
x^5 ..... 5th power,
x^6 ..... 6th power, &c.

Also, suppose $-2x$, a quantity to be involved.

Then $4x^2$ will be its square,
$-8x^3$ ..... cube,
$16x^4$ ..... 4th power,
$-32x^5$ ..... 5th power, &c.

Also, suppose $-\frac{2x}{3a}$ a quantity to be involved.

Then $\frac{4x^2}{9a^2}$ will be its square,
$-\frac{8x^3}{27a^3}$ ..... cube,
$\frac{16x^4}{81a^4}$ ..... 4th power, &c.

Suppose $a+y$, to be the root or quantity to be involved.

$$\begin{array}{l} a+y \\ a+y \\ \hline a^2+ay \\ \quad ay+y^2 \\ \hline a^2+2ay+y^2 \text{ the square,} \\ a+y \\ \hline a^3+2a^2y+ay^2 \\ \quad a^2y+2ay^2+y^3 \\ \hline a^3+3a^2y+3ay^2+y^3 \text{ the cube,} \\ a+y \\ \hline a^4+3a^3y+3a^2y^2+ay^3 \\ \quad a^3y+3a^2y^2+3ay^3+y^4 \\ \hline a^4+4a^3y+6a^2y^2+4ay^3+y^4 \text{ the fourth power, \&c.} \end{array}$$

EXAMPLES FOR PRACTICE.

1. Required the square of $3x$. Ans. $9x^2$.

2. Required the cube or third power of $4x^2$. Ans. $64x^6$.

3. Required the fourth power of $2ax^2$. Ans. $16a^4x^8$.

4. Required the cube of $-3a^2$. Ans. $-27a^6$.

5. Required the fifth power of $-\frac{3x}{4a}$ Ans. $-\frac{243x^5}{1024a^5}$

6. Required the sixth power of $-\frac{2a}{3a}$ Ans. $\frac{64x^6}{729a^6}$

7. Required the fifth power of $a+x$.

8. Required the fifth power of $a-x$.

SIR ISAAC NEWTON'S method of raising a binomial or residual quantity to any power whatever.

EXAMPLES.

1. Required the fourth power of $a+x$.

Here $\overline{a+x}|^4=a^4+4a^3x+6a^2x^2+4ax^3+x^4$.

1. In raising $a+x$ to the fourth power, a, which is the leading quantity, will have 4 for its index, that is, it will be a^4; this will be the first term.

2. The index, 4, will be the coefficient of the second term; the index of the leading quantity a, will *decrease* 1 in each succeeding term; and the second quantity x, will come into the second term, and its index will *increase* 1, in each succeeding term throughout the operation; therefore, the second term will be $4a^3x$.

3. The terms already found are a^4+4a^3x.

4. Multiply 4, the coefficient of the second term, by 3, the index of a; divide the product 12 by 2, the number of terms already found, and the quotient 6 will be the coefficient of the third term.

5. Then 6 is the coefficient to the third term, the index of a *decreasing* 1, and the index of x *increasing* 1, the third term will be $6a^2x^2$.

6. The terms thus found are $a^4+4a^3x+6a^2x^2$.

7. Multiply 6, the coefficient of the third term, by 2, the index of a in this term, and divide the product 12 by 3, the number of terms already found, the quotient 4 will be the coefficient of the fourth term; therefore, the fourth term will be $4ax^3$. The index of a having *decreased* 1, and the index of x having *increased* 1.

8. The terms already found are $a^4+4a^3x+6a^2x^2+4ax^3$.

9. In $4ax^3$, the index of a is 1, (every letter in its primitive state having 1 for an index,) therefore, $1\times4=4$, for a product, this product being divided by 4, the number of terms already found, gives 1 for the coefficient to the fifth term: the a in this term will vanish, and the index of x will increase as above; therefore, the fifth term will be x^4.

10. Hence, the fourth power of $a+x$ is $a^4+4a^3x+6a^2x^2+4ax^3+x^4$.

NOTE.—If the second quantity in the root, or quantity to be raised, be —, the terms will be + and — alternately; thus, if it be required to raise $a-x$, to the fourth power, it will be $+a^4-4a^3x+6a^2x^2-4ax^3+x^4$.

2. Required the fourth power of $3-y$.*

$$\overline{3-y}|^4=3^4-4.3^3y+\tfrac{1}{2}.3^2y^2-\tfrac{1}{3}.3y^3+\tfrac{1}{4}y^4;$$
$$\text{or, } 3^4-4.3^3y+6.3^2y^2-4.3y^3+y^4;$$
$$\text{or, } 81-108y+54y^2-12y^3+y^4.$$

EXAMPLES FOR PRACTICE.

1. Required the 3rd power of $x+y$.

2. Required the 5th power of $a+x$.

3. Required the 4th power of $5+x$.

4. Required the 4th power of $x+2$.

* The 3 in this example is placed in a similar situation to that of a in the first example, and is operated with in every respect the same.

EVOLUTION.

EVOLUTION, or the extraction of roots, is the reverse of Involution, and is commonly divided into three cases, as follows:—

CASE I.

To find the roots of simple quantities.

RULE.—*Extract the required root of the coefficient for the numeral part, and divide the index of the latter by 2 for the square root, by 3 for the cube, by 4 for the fourth root, &c.*

This root being annexed to the numeral part, will be the root required.

The square root, the fourth root, or any other even root of an affirmative quantity, may be either + or —.(a)

Any odd root of a quantity will have the same sign as the quantity itself.

EXAMPLES.

1. Required the square root of $16x^2$.

Here, $\sqrt{16}=4$, and $\sqrt{x^2}=x^{\frac{2}{2}}=x$;
$\therefore$ $4x$ is the root required.

2. Required the cube root of $27x^3$.

Here $\sqrt[3]{27}=3$, and $\sqrt[3]{x^3}=x^{\frac{3}{3}}=x^1=x$
$\therefore$ $3x$ is the root required.

3. Required the fourth root of $16x^8$.

Here $\sqrt[4]{16}=2$, and $\sqrt[4]{x^8}=x^{\frac{8}{4}}=x^2$
$\therefore$ $2x^2$ is the root required.

4. Required the cube root of $8x^4$.

Here $\sqrt[3]{8}=2$, and $\sqrt[3]{x^4}=x^{\frac{4}{3}}$ or $x.x^{\frac{1}{3}}$ or $x\sqrt[3]{x}$.
$\therefore$ $2x^{\frac{4}{3}}$ or $2x.x^{\frac{1}{3}}$ or $2x\sqrt[3]{x}$ is the root required.

(a) Here $\sqrt{16x^2}=+4x$ or $-4x$. For by note fourth, Case I. in Multiplication, $-4x\times-4x=16x^2$.

5. Required the square root of $\frac{5x^2y^2}{9a^4}$

Here $\sqrt{\frac{5x^2y^2}{9a^4}} = \frac{xy\sqrt{5.}}{3a^2} = \frac{xy}{3a^2}\sqrt{5.}$

EXAMPLES FOR PRACTICE.

1. Required the square root of $36x^4$. Ans. $6x^2$.
2. Required the square root of $5x^2y^2$. Ans. $xy\sqrt{5.}$
3. Required the cube root of $-27x^6$. Ans. $-3x^2$.
4. Required the square root of $\frac{4x^2}{9a^4}$ Ans. $\frac{2x}{3a^2}$
5. Required the fifth root $-32x^{10}y^5$. Ans. $-2x^2y$.

CASE II.

To find the square or cube root of a compound quantity.

RULE.—*Arrange the quantities according to their dimensions, as in Case III. in Division, and proceed as in Arithmetic.*

EXAMPLES.

1. Required the square root of $x^4-4x^3a+6x^2a^2-4xa^3+a^4$.

$$
\begin{array}{l|l}
 & x^4-4x^3a+6x^2a^2-4xa^3+a^4 \,(x^2-2xa+a^2 \\
 & x^4 \\
\hline
2x^2-2xa & -4x^3a+6x^2a^2 \\
 & -4x^3a+4x^2a^2 \\
\hline
2a^2-4xa+a^2 & 2x^2a^2-4xa^3+a^4 \\
 & 2x^2a^2-4xa^3+a^4 \\
\hline
\end{array}
$$

Any even root of a negative quantity is impossible; for $+x\times+x$ will not make $-x^2$; nor will $-x\times-x$ make $-x^2$. But the square, or any root of a quantity, may be nominally extracted, by placing over it its proper sign; the square root of $-3x^2$, is $\sqrt{-3x^2}$; the cube root of $-3x^2$ is $\sqrt[3]{-3x^2}$, and so on.

2. Required the cube root of $a^3+3a^2x+3ax^2+x^3$.

BY HILL'S RULE.

The cube of your first period take, &c.

$$a^3+3a^2x+3ax^2+x^3(a+x$$
$$a^3$$
$$3a^2)3a^2x+3ax^2+x^3$$
$$3a^2x$$
$$+3ax^2$$
$$+x^3$$
$$3a^2x+3ax^2+x^3$$

EXAMPLES FOR PRACTICE.

1. Required the square root of $x^2-2xy+y^2$.
Ans. $x-y$.

2. Required the square root of $9+6x+x^2$.
Ans. $3+x$.

3. Required the cube root of $27+27x+9x^2+x^3$.
Ans. $3+x$.

4. Required the cube root of $x+3x^{\frac{2}{3}}y^{\frac{1}{3}}+3x^{\frac{1}{3}}y^{\frac{2}{3}}+y$.
Ans. $x^{\frac{1}{3}}+y^{\frac{1}{3}}$

CASE III.

To find the roots of powers in general.

RULE.—1. *Arrange the terms as directed in the third case of division, then find the required root of the first term, and place it in the quotient.*

2. *Raise this root to the power of the first term, and subtract it from the first term.*

3. *Bring down the second term for a dividend.*

4. *Divide this dividend, or second term, by twice the root last found, for the square root; by three times the*

square of the root last found, for the cube root; by four times the cube of the root last found, for the fourth root, &c. and the quotient thus found will be the second term in the quotient or root.

5. *Involve the whole quotient or root thus found to the power to be extracted, then subtract this power from the given quantity of which the root is to be extracted.*

6. *Divide the greatest power of the remainder by the divisor already found, and proceed as above until the whole of the terms have been brought into operation.*

EXAMPLES.

1. Required the square root of $x^4+4x^3y+6x^2y^2+4xy^3+y^4$.

$x^4+4x^3y+6x^2y^2+4xy^3+y^4(x^2+2xy+y^2$
x^4

$2x^2)4x^3y$

$x^4+4x^3y+4x^2y^2 = (x^2+2xy)^2$

$2x^2)2x^2y^2$

$x^4+4x^3y+6x^2y^2+4xy^3+y^4=(x^2+2xy+y^2)^2$

2. Required the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

$x^6-6x^5+15x^4-20x^3+15x^2-6x+1(x^2-2x+1$
x^6

$3x^4)-6x^5$

$x^6-6x^5+12x^4-8x^3 = (x^2-2x)^3$

$3x^4)3x^4$

$x^6-6x^5+15x^4-20x^3+15x^2-6x+1=(x^2-2x+1)^3$

3. Required the fifth root of $x^5-5x^4y+10x^3y^2-10x^2y^3+5xy^4-y^5$.

$x^5-5x^4y+10x^3y^2-10x^2y^3+5xy^4-y^5(x-y$
x^5

$5x^4)-5x^4y$

$x^5-5x^4y+10x^3y^2-10x^2y^3+5xy^4-y^5=(x-y)^5$

EXAMPLES FOR PRACTICE.

1. Required the cube root of $x^3+6x^2y+12xy^2+8y^3$.
Ans. $x+2y$.

2. Required the fourth root of $y^4-4y^3+6y^2-4y+1$.
Ans. $y-1$.

3. Required the fifth root of $x^5+5x^4+10x^3+10x^2+5x+1$. Ans. $x+1$.

4. Required the fourth root of $x^{\frac{4}{3}}-4xy^{\frac{1}{3}}+6x^{\frac{2}{3}}y^{\frac{2}{3}}-4x^{\frac{1}{3}}y+y^{\frac{4}{3}}$ Ans. $x^{\frac{1}{3}}-y^{\frac{1}{3}}$

RATIOS.

RATIO is the relation which one quantity bears to another in respect of magnitude, the comparison being made by considering what multiple, part, or parts, one quantity is of the other. The first quantity being called the antecedent, and the latter the consequent. Thus, the ratio of 2 to 1 is double ratio, that of 3 to 1 triple, &c.

The two quantities that are compared are called the terms of the ratio, as 6 and 2; the first of these, 6, being called the antecedent, and the latter, 2, the consequent. Also, the index, or exponent of the ratio, is the quotient of the two terms, so the index of the ratio of 6 to 2 is $6\div 2$ or 3, which is called triple ratio.

When two terms of a ratio are equal, the ratio is that of equality; as of 3 to 3, whose index or quotient is 1, denoting the single or equal ratio. But when the terms are not equal, as of 6 to 2, it is a ratio of inequality.

When the antecedent is the greater term, as in 6 to 2, it is said to be a ratio of greater inequality; but when the antecedent is the less term, as in the ratio of 2 to 6, it is said to be a ratio of less equality.

COMPOUND RATIO.

Compound Ratio is that which is made up of two or more other ratios, by multiplying the exponents together, and so producing the compound ratio of the product of all the antecedents to the product of all the consequents. Thus, the compound ratio of 5 to 3, and 7 to 4, is the ratio of 35 to 12.

If a ratio be compounded of two equal ratios, it is called the duplicate ratio; if of three equal ratios, the triplicate ratio; if of four equal ratios, the quadruplicate, &c. The simple ratio of 3 to 2 are thus:— the duplicate ratio 9 to 4, the triplicate ratio 27 to 8, the quadruplicate ratio 81 to 16, &c.

Ratio is sometimes confounded with proportion, but very improperly, as being quite different things; for proportion is the similitude, or equality, or identity of two ratios. So the ratio of 6 to 2 is the same as that of 3 to 1, and the ratio of 15 to 5 is that of 3 to 1 also; therefore the ratio of 6 to 2 is similar or equal, or the same with that of 15 to 5, which constitutes proportion, and is thus expressed, as 6 is to 2, so is 15 to 5, or thus, 6 : 2 : : 15 : 5. So that ratio exists between two terms, but proportion between two ratios, or four terms.

PROPORTIONAL THEOREMS,

THE demonstrations of which may be found in the fifth book of Euclid.

1. Four quantities are proportionals, when the first is the same multiple of the second, that the third is of the fourth. Thus, if a, b, c, and d, be the four proportionals, they are placed thus, $a : b :: c : d$, and are read, a is to b as c to d; therefore $\frac{a}{b} = \frac{c}{d}$

2. When four quantities are proportionals, the product of the means is equal to the product of the extremes; that is, if $a : b :: c : d$, then will $ad = bc$. Also, if $a : b :: b : c$, then will $ac = b^2$. Whence it follows, that if the product of two quantities be equal to the product of two other quantities, the four quantities will be proportionals, thus, $ad = bc$, and this is turned into a proportion by making the terms of one product the extremes, and the terms of the other product the means.

3. If four quantities be proportionals, the sum of the first and second is to the second, as the sum of the third and fourth is to the fourth. Thus, if $a:b::c:d$, then will $a+b:b::c+d:d$.

4. If four quantities be proportionals, the difference between the first and second is to the second, as the difference between the third and fourth is to the fourth. Thus, if $a:b::c:d$, then will $a-b:b::c-d:d$.

5. If four quantities be proportionals, the sum of the first and second is to their difference, as the sum of the third and fourth is to their difference. Thus, if $a:b::c:d$, then will $a+b:a-b::c+d:c-d$.

6. If four quantities be proportionals, the difference between the first and second is to their sum, as the difference between the third and fourth is to their sum; that is, $a-b : a+b :: c-d : c+d$.

7. If four quantities be proportionals, they are proportionals when taken alternately. Thus, if $a:b::c:d$, then will $b:a::d:c$.

8. If a number of quantities be proportionals, the antecedent is to its consequent, as the sum of all the antecedents is to the sum of all the consequents. Thus, if $a:b::c:d::x:y::r:s$, then will $a:b::a+c+x+r:b+d+y+s$.

9. If four quantities be proportionals, the first and second may be multiplied or divided by any quantity, as well as the third and fourth, and the resulting quantities will retain the same ratio. Thus, if $a:b::c:d$, then will $\frac{a}{r}:\frac{b}{r}::cx:dx$.

10. When four quantities are proportionals, the first and third may be multiplied or divided by any quantity, as well as the second and fourth, and the resulting quantities will be proportionals. Thus, if $a:b::c:d$, then $ax:\frac{b}{s}::cx:\frac{d}{s}$

11. If the corresponding terms of two or more ranks of proportionals be multiplied together, their products will be proportionals.

Thus, if $a:b::c:d$
$e:f::g:h$
$i:j::k:l$
Then will $aei:bfj::cgk::dhl$.

12. When four quantities are proportionals, the same roots or powers of these quantities will be proportionals. Thus, if $a:b::c:d$,

Then will $a^{\frac{1}{2}}:b^{\frac{1}{2}}::c^{\frac{1}{2}}:d^{\frac{1}{2}}$ or $\sqrt{a}:\sqrt{b}::\sqrt{c}:\sqrt{d}$

Also, $a^{\frac{1}{3}}:b^{\frac{1}{3}}::c^{\frac{1}{3}}:d^{\frac{1}{3}}$ or $\sqrt[3]{a}:\sqrt[3]{b}::\sqrt[3]{c}:\sqrt[3]{d}$.

Also, $a^2:b^2::c^2:d^2$ or $a^3:b^3::c^3:d^3$.

REDUCTION OF EQUATIONS.

DEFINITIONS.

1. When two quantities are connected together by the sign of equality, the whole is called an *equation.* This equation is a proposition, and denotes the equality of two quantities or things expressed algebraically.

2. A simple equation is one which, when reduced into its simplest form, contains only the first power of the unknown quantity. Thus $x+5=9$ is an equation, expressing, that if 5 be added to x, the sum will be equal to 9. In this case, $x+5$ being equal to 9, the value of x must be 4.

Also, if $x-7=0$, this denotes, that if 7 be subtracted from x, nothing will remain; therefore, the positive part of the expression, which is x, is equal to the negative part 7.

Known quantities are represented by numbers, or the leading letters of the alphabet, and the unknown quantities by x, y, and z, or any other letter.

3. The resolution of equations is finding some quantity or quantities, from the given quantities, that will answer the conditions of the question.

4. *Five axioms* deduced from Euclid's Elements.

1. If equal quantities be added to equal quanties, their sums will be equal.
2. If equal quantities be subtracted from equal quantities, their remainders will be equal.—

Cor. Hence, if the same quantity be added to, and subtracted from any given quantity, its value will not be altered.

3. If equal quantities be multiplied by equal quantities, their products will be equal.
4. If equal quantities be divided by equal quantities, their quotients will be equal.

Cor. Hence, any quantity may be multiplied and divided by any other quantity without altering its value.

5. Things which are equal to the same things, are equal to each other.

5. In the solution of equations all the unknown quantities must be transposed to one side of the equation, and all the known quantities to the other; then proceed as in Addition. Thus, in the equation $3x+4=a+x$, the $3x$ and x are considered unknown quantities, and the 4 and the a are known quantities. (See the transposition in the next definition.)

6. When any quantity is transposed from one side of the equation to the other, its sign must be changed. Thus, if $3x+7=28$; by transposing the known quantity to the other side of this equation it becomes $3x=28-7$, then $3x=21$; therefore (*axiom* 4) $x=\frac{21}{3}=7$. Hence, transposing the 7, in this case, is only subtracting it from each side of the equation. Thus, if 7 be subtracted from $3x+7$, $3x$ will remain, and if 7 be subtracted from 28, 21 will remain; then (*axiom* 2), $3x=21$; and (*axiom* 4), $x=7$.

Also, if $3x+4=a+x$; then by transposition $3x-x=a-4$, or $2x=a-4$; $\therefore$ (*axiom* 4) $x=\frac{a+x}{2}$.

Also, if $3x-4=2x+3$, then by transposition, $3x-2x=3+4$, or $x=7$. In this case, transposing the 4 is the same as adding 4 to each side of the equation, and transposing the $2x$ is subtracting it from each side of the equation.

Cor. Hence, if all the signs in an equation be changed, the two sides will remain equal, because in this case every term is transposed.

7. If the same or equal quantities be found on each side of the equation, with like signs, they may be omitted in the transposition. Thus, if $5x+3=4a+3$, here 3 is found with the same sign on each side of the

equation, then will $5x=4a$, (3 being omitted in the transposition) and (*by axiom* 4), $x=\frac{4a}{5}$.

8. If every term in an equation be multiplied by any number, the equation will retain its equality, (*axiom* 3). Thus, if $3x+4=16$, multiply the whole by x, and it becomes $3x^2+4x=16x$.

9. If every term in any equation be divided by the same number, the equation will retain its equality (*axiom* 4). Thus, if $3x^2+4x=16x$; here, x being common to each term, the whole may be divided by x, and the equation becomes $3x+4=16$.

10. If any of the terms in an equation be a fraction, the equation may be cleared of the fractional part by multiplying the whole equation by the denominator of the fractional part; or if there be two or more fractional parts, the whole equation must be multiplied by the product of the denominators, or their least common multiple. Thus, if $3x+\frac{x}{4}=13$, multiplying the whole of this equation by 4, and it becomes $12x+x=52$, or $13x=52$; $\therefore$ (*axiom* 4) $x=4$.

Also, if $\frac{2x}{3}+\frac{x}{4}+\frac{x}{6}=13$; multiply the whole of this equation by 72, the product of all the denominators, and it becomes $48x+18x+12x=936$, or $78x=936$; $\therefore$ $x=12$. But if the whole equation be multiplied by 12, the least common multiple of 3, 4, and 6, it becomes $8x+3x+2x=156$, or $13x=156$, $\therefore x=12$.

Also, if $\frac{a}{a}+\frac{ax}{b}=\frac{y}{c}$; multiplying every term of this equation by $a\,b\,c$, the product of the denominators, it becomes $b\,c\,x+a^2c^2=a\,b\,y$.

11. If each side of an equation be squared, cubed, or raised to any other power, the results will be equal. Thus, if $x=4$, then, by squaring each side, $x^2=16$, or by cubing each side, $x^3=64$, and so on for any other power.

Also, if $x+y=4$, then by squaring each side, $x^2+2xy+y^2=16$; or, by cubing each side, $x^3+3x^2y+3xy^2+y^3=64$, &c.

12. If equal roots be extracted on each side of any equation, the roots will be equal. Thus, if $x^2=16$, extracting the square root of each side, $x=4$.

If $x^3=64$, extracting the cube root, $x=4$.

If $x^2+2xy+y^2=16$, extracting the square root of each side, $x+y=4$, and so on for any other power, being only the reverse of Def. 11.

13. If one side of an equation be an irrational or surd quantity, it may be taken away by raising each side to the power denoted by the radical sign. Thus, if $\sqrt{x+4}=7$; this signifies that the square root of $x+4$ is equal to 7; squaring each side, $x+4=49$, or which is the same thing, $\sqrt{x+4}\times\sqrt{x+4}=7^2$ or $\overline{x+4}|^{\frac{1}{2}}\times\overline{x+4}|^{\frac{1}{2}}=\overline{x+4}|^{\frac{2}{2}}=x+4$; therefore $x+4=49$, (by Note 2. Case I. Multiplication).

Cor. But if a part only of one side of an equation be a surd quantity, it may be taken away by transposing the rest of the quantities to the other side of the equation, and then raising each side to the power denoted by the radical sign.

Thus, $\sqrt[3]{x+4}+2=4$; this expresses, that the cube root of $x+4$ added to 2, is equal to 4. By transposition $\sqrt[3]{x+4}=4-2=2$; cubing each side of this equation (it being the power denoted by the sign $\sqrt[3]{\ }$), $x+4=8$; that is, $\sqrt[3]{x+4}\times\sqrt[3]{x+4}\times\sqrt[3]{x+4}=8$.

In order to make this a little clearer, $\sqrt[3]{x+4}$ being the same as $\overline{x+4}|^{\frac{1}{3}}$, then by Note 2. Case I. Multiplication,

$\overline{x+4}|^{\frac{1}{3}}=2,$

$\overline{x+4}|^{\frac{1}{3}}=2,$

$\overline{x+4}|^{\frac{2}{3}}=4,$ the square,

$\overline{x+4}|^{\frac{1}{3}}=2,$

$\overline{x+4}|^{\frac{3}{3}}=8,$ the cube or third power,

$\frac{3}{3}$ being $=1$; $\therefore \overline{x+4}|^{\frac{3}{3}}=x+4$; whence $x+4=8$, as above.

Also, if $\sqrt[5]{x+2}=3$; $\therefore \sqrt[5]{x+2}$ being $=\overline{x+2}|^{\frac{1}{5}}$,

$\overline{x+2}|^{\frac{1}{5}}=3,$

$\overline{x+2}|^{\frac{1}{5}}=3,$

$\overline{x+2}|^{\frac{2}{5}}=9,$ the square,

$\overline{x+2}|^{\frac{2}{5}}=9,$

$\overline{x+2}|^{\frac{4}{5}}=81,$ the fourth power,

$\overline{x+2}|^{\frac{1}{5}}=3,$

$\overline{x+2}|^{\frac{5}{5}}=243$ the fifth power,

as above, $\overline{x+2}|^{\frac{5}{5}}=x+2$; whence $x+2=243$.

Irrational or surd quantities are such as have no exact root, the root being extracted by placing over the quantity a radical sign or fractional index. Thus, the radical sign $\surd$ for the square root, $\sqrt[3]{}$ for the cube root, &c. or $\frac{1}{2}$ for the square root, $\frac{1}{3}$ for the cube root, &c. therefore $\surd 3$ or $3^{\frac{1}{2}}$ is called an irrational number or surd quantity, because no number, either whole or fractional, can be found, which when multiplied into itself, will produce 3. This being the case with $\surd x+4$, the surd part of the equation mentioned in the Definition.

14. In some cases irrational or surd quantities require to be reduced into different forms before they can be used. This may be done by resolving or decomposing the surd into two factors, and extracting the root of that which is rational.

Thus, $\sqrt{63}=\sqrt{9}\times\sqrt{7}$ (by Note to Case III. in Multiplication), extracting the square root of 9, which is *rational*, the expression becomes $3\times\sqrt{7}$, or $3\sqrt{7}$; also, $\sqrt{24a^3x^3}$ will resolve into $\sqrt{4a^2x^2}\times\sqrt{6ax}$; extract the square root of $4a^2x^2$, the expression becomes $2ax\times\sqrt{6ax}$, or $2ax\sqrt{6ax}$.

Also, $\sqrt[3]{54a^2x^4y}$ will resolve into $\sqrt[3]{27a^3x^3}\times\sqrt[3]{2xy}$, extract the cube root of $27a^3x^3$, the expression becomes $3ax\sqrt[3]{2xy}$; also, $\sqrt{a^2b}$ will resolve into $\sqrt{a^2}\times\sqrt{b}$; extract the square root of a^2, the expression becomes $a\sqrt{b}$; also, $\sqrt{\frac{9a^3x+9a^2y}{c^2b}}$ will resolve into $\sqrt{\frac{9a^2}{c^2}}\times\sqrt{\frac{ax+y}{b}}$; extract the square root of $\frac{9a^2}{c^2}$, the expression becomes $\frac{3a}{c}\sqrt{\frac{ax+y}{b}}$.

Also, $\sqrt[4]{\frac{16a^5x^6+16a^6x^7}{81b^5c^3+x162b^6x}}$ will resolve into $\sqrt[4]{\frac{16a^4x^4}{81b^4}}\times\sqrt[4]{\frac{ax^2+a^2x^3}{bc^2+2b^2x}}$; extracting the fourth root of $\frac{16a^4x^4}{81b^4}$, (as denoted by the radical sign $\sqrt[4]{}$) it becomes $\frac{2ax}{3b}$; hence the whole expression becomes $\frac{2ax}{3b}\sqrt[4]{\frac{ax^2+a^2x^3}{bc^2+2b^2a}}$.

Cor. It is evident from the above, that before any quantity can be multiplied into a surd quantity, and the product placed under the radical sign, the quan-

tity must be first raised to the power denoted by the sign, and then multiplied into the number or quantity under the radical sign.

Thus, in the expression $3\sqrt{7}$, if it be required to multiply the 3, into the surd part $\sqrt{7}$, the 3 must be squared (as denoted by the sign) which will be 9, then reducing this 9 to its original value, 3, it becomes $\sqrt{9}$, therefore $\sqrt{9} \times \sqrt{7} = \sqrt{63}$.

Also, in the expression $3ax\sqrt[3]{2xy}$, the $3ax$ must be cubed, (as denoted by the radical sign) and it becomes $27a^3x^3$, therefore as above, $\sqrt[3]{27a^3x^3} \times \sqrt[3]{2xy} = \sqrt[3]{54a^3x^4y}$, and so on for the other expressions given in the definition.

15. When surd quantities have different radical signs or indices, they must be reduced to a common index, before their sum, difference, product, or quotient, can be found, in its most simple form, which is only reducing the indices to a common denominator, as in vulgar fractions, and then involving each quantity to the power denoted by its numerator. Thus, if it be required to reduce $\sqrt{3}$ and $\sqrt[3]{4}$, to a common index.

Here, $\sqrt{3} = 3^{\frac{1}{2}}$, and $\sqrt[3]{4} = 4^{\frac{1}{3}}$;
therefore $3^{\frac{1}{2}} = 3^{\frac{3}{6}} = (3^3)^{\frac{1}{6}} = 27^{\frac{1}{6}}$;
and $4^{\frac{1}{3}} = 4^{\frac{2}{6}} = (4^2)^{\frac{1}{6}} = 16^{\frac{1}{6}}$.

Also, if it be required to reduce $4\sqrt{x}$ and $6\sqrt[3]{x}$ to a common index.

Here, $4\sqrt{x} = 4x^{\frac{1}{2}} = 4x^{\frac{3}{6}} = 4(x^3)^{\frac{1}{6}}$ or $4\sqrt[6]{x^3}$,
and $6\sqrt[3]{x} = 6x^{\frac{1}{3}} = 6x^{\frac{2}{6}} = 6(x^2)^{\frac{1}{6}}$ or $6\sqrt[6]{x^2}$.

When a surd quantity does not admit of decomposition, it is already in its most simple form. Thus, $\sqrt{15}$ cannot be reduced lower, neither of its factors, 3 and 5, being squares.

Also, reduce $x^{\frac{1}{n}}$ and $y^{\frac{1}{m}}$ to a common index.

Here, $\frac{1}{n}$ and $\frac{1}{m}$ being reduced to a common denominator, will be $\frac{m}{mn}$ and $\frac{n}{mn}$;

Therefore, $x^{\frac{1}{n}} = x^{\frac{m}{mn}} = (x^m)^{\frac{1}{mn}}$,

and $y^{\frac{1}{m}} = y^{\frac{n}{mn}} = (y^n)^{\frac{1}{mn}}$.

16. The reduction of surd quantities, is of great utility. Thus, if it be required to find the sum and difference of $\sqrt{28}$ and $\sqrt{63}$.

Here, $\sqrt{28} = \sqrt{4} \times \sqrt{7} = 2\sqrt{7}$, and $\sqrt{63} = \sqrt{9} \times \sqrt{7} = 3\sqrt{7}$; then $2\sqrt{7} + 3\sqrt{7} = 5\sqrt{7}$ their sum, and $3\sqrt{7} - 2\sqrt{7} = \sqrt{7}$ their difference.

Also, if it be required to find the sum of the following quantities, $\sqrt[3]{16a^3}$, $\sqrt{200a^2}$, and $\sqrt{162a^2}$. By reduction $\sqrt[3]{16a^3} = 2a\sqrt{2}$, $\sqrt{200a^2} = 10a\sqrt{2}$, and $\sqrt{162a^2} = 9a\sqrt{2}$, therefore their sum will be $21a\sqrt{2}$.

Also, if it be required to find the sum and difference of $\sqrt{\frac{12x^2y}{5}}$ and $\sqrt{\frac{15x^2y}{4}}$. These may be reduced to a common denominator, by multiplying the numerator and denominator of the first quantity by 4, and the numerator and denominator of the second quantity by 5. Thus, $\sqrt{\frac{12x^2y}{5}}$ becomes $\sqrt{\frac{48x^2y}{20}}$, and $\sqrt{\frac{15x^2y}{4}}$ becomes $\sqrt{\frac{75x^2y}{20}}$; therefore, decomposing these quantities, $\sqrt{\frac{75x^2y}{20}}$ becomes $\sqrt{25x^2} \times \sqrt{\frac{3y}{20}}$ or $5x\sqrt{\frac{3y}{20}}$, and $\sqrt{\frac{48x^2y}{20}}$ becomes $\sqrt{16x^2} \times \sqrt{\frac{3y}{20}}$ or $4x\sqrt{\frac{3y}{20}}$; hence, their sum will be $9x\sqrt{\frac{3y}{20}}$, and difference $\sqrt{\frac{3y}{20}}$.

17. If it be required to multiply two surd quantities together, their radical signs or indices being common, it is done by Note to Case III. in Multiplication; but if different, as in Def. 15, they must be reduced to a common index. Thus, if it be required to multiply $\sqrt{2}$ by $\sqrt[3]{4}$;

Here, $\sqrt{2}=2^{\frac{1}{2}}=2^{\frac{3}{6}}=(2^3)^{\frac{1}{6}}=8^{\frac{1}{6}}$,

and $\sqrt[3]{4}=4^{\frac{1}{3}}=4^{\frac{2}{6}}=(4^2)^{\frac{1}{6}}=16^{\frac{1}{6}}$,

$(128)^{\frac{1}{6}}$ Product.

Also, if the product of $4\sqrt{x}$ by $\sqrt[4]{x}$ be required,

Here, $3\sqrt[4]{x}=4x^{\frac{1}{4}}$, and $4\sqrt{x}=4x^{\frac{1}{2}}=4x^{\frac{2}{4}}$;

therefore, $3x^{\frac{1}{4}}\times 4x^{\frac{2}{4}}=12x^{\frac{3}{4}}$, or $12(x^3)^{\frac{1}{4}}$, or $12\sqrt[4]{x^3}$.

18. If it be required to divide one surd quantity by another.

Thus, divide $9\sqrt{56}$ by $3\sqrt{7}$. Here, the rational part of the dividend 9, being divided by 3, the rational part of the divisor, gives 3 for the quotient.

Also, $\sqrt{56}$, the surd part of the dividend, being divided by $\sqrt{7}$, the surd part of the divisor. Thus $\sqrt{56}\div\sqrt{7}$, $\sqrt{56\div 7}=\sqrt{8}$; therefore, $3\sqrt{8}$ is the quotient required.

But if the surds be of different kinds, they must be reduced to a common index, by Def. 15. Thus, divide 3 by $\sqrt[3]{9}$.

Here, $3=9^{\frac{1}{2}}=9^{\frac{3}{6}}$, and $\sqrt[3]{9}=9^{\frac{1}{3}}=9^{\frac{2}{6}}$;

Then, $\frac{9^{\frac{3}{6}}}{9^{\frac{2}{6}}}=9^{\frac{3}{6}-\frac{2}{6}}=9^{\frac{1}{6}}$.

Also, if it be required to divide $12\sqrt[4]{y^3}$ by $4\sqrt{y}$.

Here, $12\sqrt[4]{y^3} = 12y^{\frac{3}{4}}$, and $4\sqrt{y} = 4y^{\frac{1}{2}} = 4y^{\frac{2}{4}}$;

Then, $\frac{12y^{\frac{3}{4}}}{4y^{\frac{2}{4}}} = 3y^{\frac{3}{4}-\frac{2}{4}} = 3y^{\frac{1}{4}}$, or $3\sqrt[4]{y}$.

Let $\frac{2}{3}y^{\frac{1}{n}}$ be divided by $\frac{3}{4}y^{\frac{1}{m}}$.

Here, $\frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$.

And, $y^{\frac{1}{n}} \div y^{\frac{1}{m}} = y^{\frac{m}{mn}} \div y^{\frac{n}{mn}} = y^{\frac{m-n}{mn}}$.

Whence $\frac{8}{9}y^{\frac{m-n}{mn}}$ the quotient required.

NOTE.—The division of Surds is only subtracting the index of the divisor from the index of the dividend. From this it follows, that the denominator of a fraction may be placed in the numerator, and *vice versa*, by changing the sign of its index.

1. Let $\frac{1}{y^2}$ be expressed with a negative index.

Here, by the note, $\frac{1}{y^2} = \frac{y^{-2}}{1} = y^{-2}$.

2. Let $\frac{b}{y^2}$ be expressed by a negative index.

Here, $\frac{b}{y^2} = \frac{by^{-2}}{1}$, or by^{-2}.

3. Let $\frac{1}{x+y}$ be expressed by a negative index.

Here, $\frac{1}{x+y} = \frac{1}{(x+y)^1} = (x+y)^{-1}$.

4. Let $x.(x+y)^{-\frac{1}{2}}$ be expressed by a positive index.

Here, $x.(x+y)^{-\frac{1}{2}} = \frac{x}{(x+y)^{\frac{1}{2}}}$.

SIMPLE EQUATIONS.

EXAMPLES

In Simple Equations, with their solutions, in which the preceding definitions and reductions are applied.

1. Given $5x+35=4x+38$, to find the value of x.

Here (6), by transposition, $5x-4x=38-35$;

$\therefore$ by Case II. in Addition, $x=3$.

2. Given $3x+7-2x=6x-13$, to find the value of x.

Here (6), by transposition, $7+13=6x+2x-3x$;

$\therefore$ by Case II. in Addition, $20=5x$;

then (by Axiom 4), $x=\frac{20}{5}=4$.

Or thus,

(6) by transposition, $3x-2x-6x=-13-7$.

$\therefore$ By Addition, $-5x=-20$;

then (6. Cor.), $5x=20$; $\therefore$ (ax. 4) $x=\frac{20}{5}=4$.

3. Given $3a+4x=2b-x$, to find the value of x.

Here (6), by transposition, $4x+x=2b-3a$,

or $5x=2b-3a$; $\therefore$ (ax. 4.) $x=\frac{2b-3a}{5}$.

N. B.—From these solutions it appears, that the unknown quantities may be transposed to either side of the equation, and the known quantities to the other.

NOTE.—The Student may prove his questions by substituting the value of x, when found, in the given equation, and if the sides remain equal after substitution, the number substituted will be the value of the unknown quantity; thus, in the first example, x is found to be equal to 3, then will $5x=15$, and $4x=12$; $\therefore$ if 15 be substituted for $5x$, its value, in the given equation, and 12 for $4x$, its value, then the equation, which is $5x+35=4x+38$, becomes $15+35=12+38$, or $50=50$; therefore, the value of x, in this case, is 3.

4. Given $3x-4=\frac{x}{2}+6$, to find the value of x.

Here (10), multiplying the equation by 2, the denominator of the fractional part, the equation becomes $6x-8=x+12$;

$\therefore$ by transposition, $6x-x=12+8$;

then, Case II. Addition, $5x=20$;

$\therefore$ (by Axiom 4,) $x=\frac{20}{5}=4$.

5. Given $4x+\frac{x}{5}=\frac{3x}{3}+16$, to find the value of x.

Here (10), multiplying the equation by 15, the least common multiple of 5 and 3,

$$60x+3x=15x+240;$$

$\therefore$ (6.) by transposition, $60x+3x-15x=240$;

then, by Addition, $48x=240$;

whence (Ax. 4.) $x=\frac{240}{48}=5$.

6. Given $4ax+6ab=2cx+7a$, to find the value of x

Here (6), by transposition, $4ax-2cx=7a-6ab$;

$\therefore$ (Ax. 4.) $x=\frac{7a-6ab}{4a-2c}$.

7. Given $\frac{x}{a}+\frac{x}{b}+\frac{4}{c}=\frac{8}{d}+4$, to find the value of x.

Here (10), multiplying the equation by $abcd$, the product of the denominators, it becomes,

$$bcdx+acdx+4abd=8abc+4abcd;$$

by transposition, $bcdx+acdx=8abc+4abcd-4abd$;

$\therefore$ (Ax. 4.) $x=\frac{8abc+4abcd-4abd}{bcd+acd}$, or $\frac{4ab}{cd}\left(\frac{2c+cd-d}{b+a}\right)$,

(by decomposition).

8. Given $4x+\frac{3x-4}{5}=7+\frac{x+15}{3}$, to find the value of x.

Here (10), multiplying the equation by 15, the least common multiple of 3 and 5,

$$60x+9x-12=105+5x+75;$$

by transposition, $60x+9x-5x=105+12+75$,

or $64x=192$; $\therefore$ (Ax. 4,) $x=\frac{192}{64}=3$.

9. Given $42+\frac{12x-12}{16}=\frac{3x+1}{8}+\frac{10x+6}{2}+15$, to find the value of x.

Here, multiplying the equation by 16, the least common multiple of 16, 8, and 2, the equation becomes,

$$672+12x-12=6x+2+80x+48+240;$$

by transposition, $672-12-2-48-240=80x+6x-12x$, or $370=74x$;

$$\therefore\ x=\frac{370}{74}=5.$$

10. Given $3x-\frac{x-2}{2}=16$, to find the value of x.

Multiplying the equation by 2, $6x-x+2^{*}=32$;

by transposition, $6x-x=32-2$;

or $5x=30$; $\therefore\ x=\frac{30}{5}=6$.

NOTE.—It may not be improper to remark why the 2, with the asterisk over it, becomes +, when cleared of fractions. The reason is obvious, for in clearing the equation of the fraction, the x, which is +, becomes minus, therefore, the 2 must be made plus in order to obtain the difference between the x and the 2, as required in the given equation.

It may also be further remarked, that if any fraction, having a compound quantity for its numerator, be a minus quantity, all the signs in the numerator must be changed when the equation is cleared of fractions, as in the eleventh and twelfth examples.

11. Given $4x-\frac{x+4-1}{3}=15\frac{1}{2}-\frac{3x+2}{2}$, to find the value of x.

Here (10), multiplying the equation by 6, the least common multiple of 3, 2, and 2,

$$24x-2x-8+2=93-9x-6;$$

by transposition, $24x-2x+9x=93+8-2-6$,

or $31x=93$; $\therefore\ x=\frac{93}{31}=3$.

12. Given $3x-\frac{ax+b-c+d}{r}=n$, to find the value of x.

Here, multiplying the equation by r,

$$3rx-ax-b+c-d=nr;$$

by transposition, $3rx-ax=nr+b-c+d$;

$$\therefore\ x=\frac{nr+b-c+d}{3r-a}.$$

13. Given $\frac{12y-9}{9}+\frac{7y-13}{2y-8}=\frac{4y}{3}+4$, to find the value of y.

Multiplying the equation by 9,

$$12y-9+\frac{63y-117}{2y-8}=12y+36;$$

$$\therefore\ \text{(6 and 7,)}\ \frac{63y-117}{2y-8}=36+9=45;$$

Multiplying this equation by $2y-8$,

$$63y-117=90y-360;$$

by transposition, $360-117=90y-63y$,

or $243=27y$; $\therefore\ y=\frac{243}{27}=9$.

14. Given $\frac{3y+6}{6}+\frac{8y-23}{5y-12}=\frac{9y+13}{18}$, to find the value of y.

Multiplying the equation by 18,

$$9y+18+\frac{144y-414}{5y-12}=9y+13;$$

by transposition, $\frac{144y-414}{5y-12}=-5$;

Multiplying this equation by $5y-12$,

$$144y-414=-25y+60:$$

by transposition, $144y+25y=414+60$,

or $169y=474$;

Hence, $y=\frac{474}{169}=2\frac{136}{169}$.

15. Given $\frac{10y+4}{4}:\frac{36-2y}{6}::6:4$, to find the value of y.

Here, by Theo. 2, $\frac{10y+4}{4}\times 4=\frac{36-2y}{6}\times 6$,

or $10y+4=36-2y$;

by transposition, $10y+2y=36-4$,

or $12y=32$; hence, $y=\frac{32}{12}=2\frac{2}{3}$.

16. Given $\frac{3y+4}{5}+4:\frac{y+7}{3}-2::12:2$, to find the value of y.

Here, by Theo. 2, $\left(\frac{3y+4}{5}+4\right).2=\left(\frac{y+7}{3}-2\right).12$,

or, $\frac{6y+8}{5}+8=4y+28-24$;

by transposition, $\frac{6y+8}{5}=4y-4$;

$\therefore$ multiplying by 5, $6y+8=20y-20$;
by transposition, $8+20=20y-6y$; or $28=14y$;
hence, $y=\frac{28}{14}=2$.

17. Given $\sqrt{9y+13}=7$, to find the value of y.

Here (13), squaring each side of the equation,
$9y+13=49$; by transposition, $9y=49-13$,
or $9y=36$; hence $y=\frac{36}{9}=4$.

18. Given $\sqrt[3]{4y+36}-2=2$, to find the value of y.

Here (Cor. to 13), $\sqrt[3]{4y+36}=2+2=4$;
$\therefore$ (13) cubing each side, $4y+36=64$;
by transposition, $4y=64-36=28$;
hence, $y=\frac{28}{4}=7$.

19. Given $\sqrt[5]{y+988}+2=6$, to find the value of y.

Here (Cor. to 13), $\sqrt[5]{y+988}=6-2=4$;
$\therefore$ (13) by raising each side to the 5th power,
$y+988=1024$;
by transposition, $y=1024-988=36$.

20. Given $\sqrt{4y-11}=2\sqrt{y}-1$, to find the value of y.

Here, squaring each side of the equation,
$4y-11=4y-4\sqrt{y}+1$
then (by 7, and transposition,) $4\sqrt{y}=11+1=12$;
$\therefore \sqrt{y}=\frac{12}{4}=3$; hence, (13.) $y=9$.

21. Given $\sqrt{y+16}=2+\sqrt{y}$, to find the value of y.

Squaring each side, $y+16=4+4\sqrt{y}+y$;

then (by 7, and transposition,) $16-4=4\sqrt{y}$,

or $12=4\sqrt{y}$, $\therefore$ (dividing by 4) $3=\sqrt{y}$;

hence (13), squaring each side, $y=9$.

22. Given $\sqrt{9}\times\sqrt{y+3\frac{1}{5}}=\sqrt{9y}+2$, to find the value of y.

Here (17,) $\sqrt{9y+28}=\sqrt{9y}+2$;

squaring each side, $9y+28=9y+4\sqrt{9y}+4$;

by (7) $28=4\sqrt{9y}+4$;

by transposition, $28-4=4\sqrt{9y}$, or $24=4\sqrt{9y}$;

$\therefore$ $\sqrt{9y}=\frac{24}{4}=6$; squaring each side, $9y=36$;

hence, $y=\frac{36}{6}=6$.

23. Given $a+y=b-\sqrt{2ay+y^2}$, to find the value of y.

Here, by transposition, $\sqrt{2ay+y^2}=b-a-y$;

squaring each side, $2ay+y^2=b^2-2ab+a^2+2ay+y^2-2by$;

$\therefore$ (by 7, and transposition,) $2by=b^2-2ab+a^2$;

$\therefore$ $y=\frac{b^2-2ab+a^2}{2b}=\frac{\overline{b-a}|^2}{2b}$.

24. Given $\frac{\sqrt{y}+34}{\sqrt{y}+1}=\frac{\sqrt{y}+46}{\sqrt{y}+2}$, to find the value of y.

Here (10), multiplying the equation by $(\sqrt{y}+1)$. $(\sqrt{y}+2)$, or which is the same thing, multiply the

numerator of the first part of the equation by the denominator of the second part, and the numerator of the second part by the denominator of the first part, omitting the denominators,

$$(\sqrt{y}+34).(\sqrt{y}+2)=(\sqrt{y}+46).(\sqrt{y}+1),$$

$$\text{or } y+36\sqrt{y}+68=y+47\sqrt{y}+46;$$

(by 7 and transposition,) $68-46=47\sqrt{y}-36\sqrt{y}$;

$$\text{or } 22=11\sqrt{y}; \quad \therefore \sqrt{y}=\frac{22}{11}=2;$$

$\therefore$ (13), squaring each side, $y=4$.

25. Given $\frac{3y-9}{\sqrt{3y}+3}+1=\frac{\sqrt{3y}+9}{3}+5$, to find the value of y.

Since $3y-9=(\sqrt{3y}-3).(\sqrt{3y}+3)$, the equation becomes $\frac{(\sqrt{3y}-3).(\sqrt{3y}+3)}{\sqrt{3y}+3}+1=\frac{\sqrt{3y}+9}{3}+5$;

then, by cancelling the like quantities in the numerator and the denominator of the first part of this equation,

$$\sqrt{3y}-3+1=\frac{\sqrt{3y}+9}{3}+5$$

$\therefore$ (10) $3\sqrt{3y}-9+3=\sqrt{3y}+9+15$;

by transposition, $3\sqrt{3y}-\sqrt{3y}=9+15+9-3$,

$$\text{or } 2\sqrt{3y}=30; \quad \therefore \sqrt{3y}=15;$$

squaring each side, $3y=225$; hence $y=\frac{225}{3}=75$.

26. Given $\sqrt{13y+y\sqrt{4y+12}}=7y^{\frac{1}{2}}$, to find the value of y.

Here (13), squaring each side, $13y+y\sqrt{4y+12}=49y$;

by transposition, $y\sqrt{4y+12}=36y$; dividing by y, $\sqrt{4y+12}=36$; squaring each side, $4y+12=1296$; $\therefore\ 4y=1296-12=1284$;

hence, $y=\frac{1284}{4}=321$.

27. Given $\sqrt{7+y}+\sqrt{y}=\frac{28}{\sqrt{7+y}}$ to find the value of y.

Here (10), $\sqrt{7+y}\times\sqrt{7+y}+\sqrt{y}\times\sqrt{7+y}=28$,

or $7+y+\sqrt{7y+y^2}=28$;

by transposition, $\sqrt{7y+y^2}=21-y$;

squaring each side, $7y+y^2=441-42y+y^2$;

by transposition, $7y+42y=441$;

or $49y=441$; $\therefore\ y=9$.

28. Given $\sqrt[4]{a+b}=\sqrt[5]{\frac{2a+2b}{y}}$, to find the value of y.

Here (13), raising each side to the fifth power,

$$(a+b).\sqrt[4]{a+b}=\frac{2a+2b}{y},$$

$\therefore$ (ax. 4.) dividing the equation by $a+b$,

$$\sqrt[4]{a+b}=\frac{2}{y};\ \therefore\ y\sqrt[4]{a+b}=2;$$

whence, $y=\frac{2}{\sqrt[4]{a+b}}$.

EXAMPLES FOR PRACTICE.

1. Given $4x+2=22$, to find the value of x.
Ans. 5.

2. Given $3x-2=16$, to find the value of x.
Ans. $x=6$.

3. Given $2x-a=b$, to find the value of x.
Ans. $x=\frac{a+b}{2}$.

4. Given $18x+13=59-5x$, to find the value of x.
Ans. $x=2$.

5. Given $4x-4+\frac{x}{3}=62-3x$, to find the value of x.
Ans. $x=9$.

6. Given $\frac{x}{4}+7=\frac{x}{6}+8$, to find the value of x.
Ans. $x=12$.

7. Given $4x^2-7x=8x-x^2$, to find the value of x.
Ans. $x=3$.

8. Given $ax^2-abx=acx$, to find the value of x.
Ans. $x=b+c$.

9. Given $\frac{4x+4}{5}+3x=\frac{8+3x}{5}+12$, to find the value of x.
Ans. $x=4$.

10. Given $\frac{5x+2}{8}+2x=8+\frac{6x-4}{4}$, to find the value of x.
Ans. $x=6$.

11. Given $\frac{4x+2}{4}+\frac{6x-7}{5}=8+\frac{8x-28}{8}$, to find the value of x.
Ans. $x=4\frac{1}{2}$.

12. Given $\frac{3x+9}{6}+\frac{21-x}{3}=\frac{5x-9}{18}+8$, to find the value of x.
Ans. $x=9$.

13. Given $x-\frac{3x-9}{9}=\frac{4x-3}{5}$, to find the value of x.

Ans. $x=12$.

14. Given $4\frac{1}{4}+\frac{2x+1}{4}=13-\frac{3x-15}{5}$, to find the value of x. Ans. $x=10$.

15. Given $\frac{25+5x}{9}+18=6x-\frac{2-4x}{7}+\frac{3x-8}{4}$, to find the value of x. Ans. $x=3\frac{697}{1705}$.

16. Given $5x-\frac{2x-6}{5}-\frac{3x+6}{6}=\frac{5x+20}{15}+29$, to find the value of x. Ans. $x=8$.

17. Given $5\frac{1}{12}-\frac{4x-5}{3}-\frac{24+5x}{12}=\frac{32-10x}{4}+x-\frac{4x+7}{6}$, to find the value of x. Ans. $x=5$.

18. Given $\frac{5x-2}{2}-\frac{4x-18}{3}-4\frac{1}{4}=6\frac{1}{2}+\frac{x-1}{4}$, to find the value of x. Ans. $x=6$.

19. Given $10-\frac{4x-8}{6}-\frac{12-x}{2}=7\frac{1}{8}-\frac{2x+9}{8}$, to find the value of x. Ans. $x=8$.

20. Given $\frac{4+x}{x}=7-\frac{5x+5}{x}$, to find the value of x.

Ans. $x=9$.

21. Given $\frac{3a+x}{a}+b=\frac{ax+4}{c}$, to find the value of x.

Ans. $x=a.\frac{(4-bc-3c)}{c-a^2}$.

22. Given $\frac{1}{2}ax+\frac{2}{3}bc-\frac{3}{4}a=\frac{5}{6}ax+2ab+c$, to find the value of x. Ans. $x=\frac{8bc-9a-24ab-12c}{4a}$.

23. Given $\frac{ax}{b}+\frac{bx}{c}+\frac{dx}{a}-r=4+(a+b).x$, to find the value of x. Ans. $x=.\frac{abc.(4+r)}{ac(a-ab-b^2)+b.(ab+cd)}$.

24. Given $\frac{5x+10}{25}+\frac{3x+4}{2x-4}=\frac{x}{5}+4\frac{2}{5}$, to find the value of x. Ans. $x=4$.

25. Given $\frac{12x+9}{18}-\frac{13x-2}{14x-20}=\frac{6x-8}{9}$, to find the value of x. Ans. $x=4$.

26. Given $\frac{14x-28}{28}+\frac{5x+8}{3x-20}=\frac{7x-28}{14}+13$, to find the value of x. Ans. $x=8$.

27. Given $\sqrt{\frac{4x}{7}}=4$, to find the value of x. Ans. $x=28$.

28. Given $\sqrt{4x+16}=6$, to find the value of x. Ans. $x=5$.

29. Given $\sqrt{8x+1}+3=10$, to find the value of x. Ans. $x=6$.

30. Given $\sqrt[3]{4x-4}+5=7$, to find the value of x. Ans. $x=3$.

31. Given $\sqrt[4]{2x+6}+8=10$, to find the value of x. Ans. $x=5$.

32. Given $\sqrt[5]{7x+201}-1=2$, to find the value of x. Ans. $x=6$.

33. Given $\sqrt{x+7}=7-\sqrt{x}$, to find the value of x. Ans. $x=9$.

34. Given $\sqrt{x+9}=1+\sqrt{x}$, to find the value of x. Ans. $x=16$.

35. Given $\sqrt{4x+9}=9-\sqrt{4x}$, to find the value of x. Ans. $x=4$.

36. Given $\sqrt{5x^2+9}=2x+\sqrt{9}$, to find the value of x. Ans. $x=12$.

37. Given $\sqrt{9x+37}=3\sqrt{x}+1$, to find the value of x. Ans. $x=36$.

38. Given $\frac{10y+5}{3} : \frac{3y+4}{5} :: 15 : 3$, to find the value of y. Ans. $y=7$.

39. Given $\frac{17-2y}{3}+3y : \frac{2y+41}{7}+5 :: 5 : 4$, to find the value of y. Ans. $y=4$.

40. Given $\frac{2y+6}{y-3} : \frac{4y-24}{3y+3} :: 6 : 4$, to find the value of y. Ans. $y=1\frac{2}{13}$.

41. Given $\frac{3y+a}{b} : \frac{2y+c}{d} :: 1 : 2$, to find the value of y. Ans. $y=\frac{bc-2ad}{6d-2b}$.

42. Given $\sqrt{3y+4}=\sqrt[4]{9y^2+112}$, to find the value of y. Ans. $y=4$.

43. Given $\sqrt[4]{5y+6}=\frac{4}{\sqrt[4]{5y+6}}$, to find the value of y. Ans. $y=2$.

44. Given $\sqrt[5]{6y+2}=\frac{\sqrt[5]{204y+4}}{2}$, to find the value of y. Ans. $y=5$.

45. Given $\sqrt[3]{11y+144}=\sqrt[4]{\frac{11y+144}{12}}$, to find the value of y. Ans. $y=144$.

46. Given $\sqrt[4]{a+b}=\sqrt[5]{\frac{a+b}{y}}$, to find the value of y. Ans. $y=\frac{1}{\sqrt[4]{a+b}}$.

47. Given $\sqrt{a^{\frac{1}{2}}-y^{\frac{1}{2}}} : \sqrt{a^{\frac{1}{2}}-b^{\frac{1}{2}}} :: \sqrt{a^{\frac{1}{2}}+b^{\frac{1}{2}}} : \sqrt{a^{\frac{1}{2}}+y^{\frac{1}{2}}}$, to find the value of y. Ans. $y=b$.

48. Given $\frac{4\sqrt{4y+4}}{5\sqrt{2y+3}}=1\frac{1}{15}$, to find the value of y. Ans. $y=3$.

49. Given $\frac{6y-16}{\sqrt{6y+4}}-1=\frac{\sqrt{6y}-4}{2}$, to find the value of y. Ans. $y=6$.

50. Given $\frac{\sqrt[3]{8y+40}}{2}=\frac{4}{\sqrt[3]{y+5}}$, to find the value of y. Ans. $y=3$.

51. Given $\frac{\sqrt[3]{27a+54y}}{\sqrt[3]{4y^2+b}}=\frac{3}{\sqrt[3]{a+2y}}$, to find the value of y. Ans. $y=\frac{b-a^2}{4a}$.

THE APPLICATION OF SIMPLE EQUATIONS TO PROBLEMS OF ONE UNKNOWN QUANTITY.

The method of solving algebraical problems or questions is to resolve them into equations, and then proceed by the general rules there given.

In order to perform this, represent the quantity required, or some other quantity in the question, from which the required quantity may be found, by x or y, or any other letter of the alphabet, then perform with this letter and the given quantities in the question, by means of the common signs, the same operations and reasonings that it would require, or be necessary to make, if the quantities were known, and it was required

to verify them, and the result will give an equation, which may be solved by the rules already laid down.

To facilitate the progress of the pupil, the following examples, with their solutions, are given in a style as familiar as the nature of the subject admits :—

1. A Lady was asked her age, who replied thus :—

My age, if multiplied by three,
Two-sevenths of that product tripled be,
The square root of two-ninths of that is four,
Now tell my age, or never see me more.

Here, the method of reasoning with her age is given in order to find it, from which it appears, that if her age could be known and operated with, as directed in the question, the *four* therein mentioned would be produced. *(See the proof.)*

$\therefore$ Let x = her age in years;

then, $3x$ = her age multiplied by 3;

also, $\frac{2}{7}\times 3x\times 3=\frac{6x}{7}\times 3=\frac{18x}{7}$ = two-sevenths of that product tripled;

and, $\frac{2}{9}\times\frac{18x}{7}=\frac{36x}{63}$, this reduced to its lowest terms $=\frac{4x}{7}$ = two ninths of $\frac{18x}{7}$;

by the question, the square root of $\frac{4x}{7}$ must be equal to 4;

that is, $\sqrt{\frac{4x}{7}}=4$, (an equation);

then, squaring each side, $\frac{4x}{7}=16$;

multiplying by 7, $4x=112$;

$\therefore x=\frac{112}{4}=28$, her age.

PROOF.

28
3
—
84, her age multiplied by 3,
2
—
7) 168
—
24, two sevenths of that product,
3
—
72, the product tripled,
2
—
9) 144
—
16, two ninths of the tripled product,
—
4, the square root of these two-ninths, as mentioned in the question;

∴ 28 answers the conditions of the question.

2. A Fish was caught, whose tail weighed 9lbs., his head weighed as much as his tail and half his body, and his body weighed as much as his head and tail. Required the weight of the fish?

It appears from the nature of this question, that to substitute x for the weight of the whole fish would not conveniently answer the conditions of the question; but if the weight of the body was known, the other parts could be easily determined, and hence, the weight of the whole fish.

∴ Let x = the weight of the body in lbs.; then $\frac{x}{2}$ = half the weight of the body, which being added to 9 lbs. (the weight of the tail) gives $9+\frac{x}{2}$ = the weight

of the head in lbs., and $9+\frac{x}{2}+9$=the weight of the head and tail, which, by the question, is equal the weight of his body;

$$\therefore\ x=9+\frac{x}{2}+9,\ \text{(an equation,)}$$

(10.) multiplying by 2, $2x=18+x+18$;

by transposition, $2x-x=18+18$,

or $x=36$ weight of the body,

and $9+\frac{x}{2}=27$, weight of the head,

9, weight of the tail,

72, weight of the fish.

Or thus:—

Let $2x$ = the weight of the body in lbs.;

then, $x+9$ = the weight of the tail;

$$\therefore\ 9+x+9=2x;$$

by transposition, $18=x$;

$\therefore$ the fish weighed $36+27+9=72$lbs.

3. What number is that, to the double of which if 14 be added, the sum will be 52?

Here the learner may say, if I knew the number, I could double it and add 14 to the product, and the sum would be equal to 52 by the question;

$$\therefore\ \text{let } x = \text{the number required;}$$

then, by the question, $2x+14=52$;

by transposition, $2x=52-14=38$;

$$\therefore\ x=\frac{38}{2}=19.$$

4. What number is that, from the double of which if 16 be subtracted, the remainder is 12?

Here, as above, let x = the required number;

then $2x-16=12$;

by transposition, $2x=12+16=28$;

$$\therefore\ x=\frac{28}{2}=14.$$

5. Said A, I am twice as old as B; to which C answered, I am as old as you both, and the sum of our respective ages is 90 years. What was each person's age?

Here, if the Student knew B's age, he would easily find the other;

$\therefore$ let x = B's age in years;

then, by the question, $2x$ = A's age,

and $3x$ = C's age;

hence, by C's answer, the sum of these three quantities is equal to 90;

$$\therefore\ x+2x+3x=90,\ \text{or}\ 6x=90;$$

$$\text{hence,}\ x=\frac{90}{6}=15,\ \text{B's age};$$

$\therefore$ A's 30, B's 15, and C's 45 years.

Or thus:—

Let x = A's age in years; then, A being twice as old as B, $\frac{x}{2}$ = B's age, and C being as old as A and B together, $x+\frac{x}{2}$ = C's age in years;

then, the sum of all their ages being 90 years,

$$x+\frac{x}{2}+x+\frac{x}{2}=90,\ \text{or}\ 3x=90;$$

$$\therefore\ x=\frac{90}{3}=30,\ \text{A's age};$$

hence, A's 30, B's 15, and C's 45 years.

6. A gentleman distributed 20s. amongst four poor people; to the second he gave twice, to the third three times, and to the fourth four times as much as the first. How much did he give to each?

Let x = the sum he gave to the first in shillings;

then $2x$ = the sum he gave to the second;

$3x$ = the sum he gave to the third;

and $4x$ = the sum he gave to the fourth;

$\therefore\ x+2x+3x+4x=20$, or $10x=20$;

$\therefore\ x=\frac{20}{10}=2$;

hence, 2s. 4s. 6s. and 8s. he gave to each respectively.

7. If the distance from Leeds to London be 192 miles, and two travellers, A and B, set out at the same time to meet each other, A from Leeds and B from London; A goes 25 miles, and B 23 miles a-day. In how many days will they meet?

Let x = the number of days required;

then $25x$ = the distance A will travel before he meets B,

and $23x$ = the distance B will travel before he meets A;

$\therefore$ these distances added together will = the distance from Leeds to London;

hence, $25x+23x=192$, or $48x=192$;

$\therefore\ x=\frac{192}{48}=4$.

8. A Wine Merchant sold 12 gallons of wine, at a certain price per gallon, and afterwards 17 gallons more at the same rate, and received £4. 5s. more at this sale than at that. Required the price per gallon.

Let x = the price per gallon in shillings;

then $12x$ = the sum received for 12 gallons,

and $17x$ = the sum received for 17 gallons;

$\therefore$ by the question, the difference of these sums is = £4. 5s. or 85 shillings;

hence $17x - 12x = 85$, or $5x = 85$;

$$\therefore x = \frac{85}{5} = 17\text{s.}$$

9. Divide a sum of money amongst A, B, C, and D, as follows: give to B 5s. more than to A, and to C 2s. more than to B, and to D 6s. less than to C; the sum to be divided is 1s. more than 8 times the sum given to A. Required each person's share.

Here, if the Student knew the sum given to A, he would readily determine the rest;

$\therefore$ let x = A's share in shillings;

then $x+5$ = B's share in shillings;

$x+7$ = C's share in shillings;

and $x+1$ = D's share in shillings;

now, the sum to be divided being one shilling more than eight times A's share;

$\therefore 8x+1$ = the sum to be divided, which, by the question, will be equal to the sum of their shares;

$$\therefore x+x+5+x+7+x+1 = 8x+1,$$

$$\text{or } 4x+13 = 8x+1;$$

by transposition, $8x - 4x = 13 - 1$,

$$\text{or } 4x = 12; \therefore x = 3;$$

whence, A's 3, B's 8, C's 10, and D's 4 shillings.

10. A and B, with each an equal number of marbles, began to play with C and D; after a certain number of games, A found that he had won 10, and that B had lost 20; A had now twice as many as B. How many had each at first?

Let x = the number each had at first;

then $x+10$ = the number A had after playing,

and $x-20$ = the number B had after playing;

$\therefore$ by the question, A's number being equal to twice B's,

$$2.(x-20)=x+10, \text{ or } 2x-40=x+10;$$

by transposition, $2x-x=10+40$; $\therefore$ $x=50$.

11. A farmer has two stacks of hay, each containing the same number of tons; from one of them he sells 7 tons, and from the other 14, and then found that twice as much remained in one as the other. Required the number of tons in each stack at first.

Let x = the number of tons in each;

then $x-7$ = the tons left in one,

and $x-14$ = the tons left in the other;

$\therefore$ by the question, $2(x-14)=x-7$,

or $2x-28=x-7$;

by transposition, $2x-x=28-7$; $\therefore$ $x=21$.

12. A farmer bought 12 sheep for £10. 12s.; for part of which he gave 23s. a head, and for the rest 15s. a head. Required the number of each.

Let x = the number at 23s. a head;

then $23x$ = their price in shillings;

also, $12-x$ = the number at 15s. a head,

and $15.(12-x)$, or $180-15x$ = their price;

$\therefore$ $23x+180-15x=212$;

by transposition, $23x-15x=212-180$,

or $8x=32$; $\therefore$ $x=4$;

whence, there were 4 at 23s. and 8 at 15s. a head.

13. What two numbers are as 4 to 5, from each of which if 10 be subtracted, the remainders will be as 2 to 3?

Let $4x$ and $5x$ denote the numbers required;

then $4x-10$, and $5x-10$ will denote the remainders;

$\therefore$ by the question, $4x-10 : 5x-10 :: 2 : 3$,

and (by Theo. 2.) $12x-30=10x-20$;

by transposition, $12x-10x=30-20$,

or $2x=10$; $\therefore\ x=5$;

whence $4x=20$, and $5x=25$; $\therefore$ the numbers are 20 and 25.

Or thus:—

Let x = the lesser number;

then $4 : 5 :: x : \frac{5x}{4}$ = the greater number;

also, by the question, $x-10$ = the first remainder,

and $\frac{5x}{4}-10$ = the second remainder;

$\therefore\ x-10 : \frac{5x}{4}-10 :: 2 : 3$;

multiplying means and extremes,

$$3x-30=\frac{5x}{2}-20;$$

$$\therefore\ 6x-60=5x-40;$$

by transposition, $6x-5x=60-40$; $\therefore\ x=20$;

whence, the numbers are 20 and 25.

14. Two merchants, A and B, entered in partnership; A advanced £50 more than B; they each gained £100, which made their respective shares in the proportion of 13 to 12. How much did each person advance?

Let x = the sum A advanced;

then $x-50$ = the sum B advanced;

$\therefore$ by the question, $x+100$ = A's share after gaining £100,

and $x-50+100$, or $x+50$ = B's share after gaining £100;

$\therefore x+100 : x+50 :: 13 : 12$;

$\therefore 12x+1200=13x+650$;

by transposition, $x=550$;

whence, A advanced £550, and B £500.

15. Between 9 and 10 o'clock in the evening, when the hour and minute hands are exactly together. Required the time.

Let x = the time past 9;

then, since the minute hand goes 12 times round whilst the other goes once,

$$12 : 1 :: 9+x : x; \therefore 12x=9+x;$$

by transposition, $11x=9$; $\therefore x=\frac{9}{11}=49' 5''\frac{5}{11}$,

the time past 9 o'clock.

16. Bought two pieces of cloth, which together measured 70 yards, for £49. 10s.; I gave 15s. per yard for one piece, and 12s. per yard for the other; this latter piece being damaged I returned it. How much had I to pay?

Let x = the number of yards at 15s. per yard;

then $70-x$ = the number of yards at 12s. per yard;

$\therefore 15x$ = the cost of the piece at 15s. per yard,

and $840-12$ = the cost of the piece at 12s. per yard;

$\therefore$ by the question, $15x+840-12x=990$;

by transposition, $3x=990-840=150$;

$$\therefore x=\frac{150}{3}=50,$$

whence, 50 yards at 15s. = £37. 10s. the sum to pay.

17. Four men, A, B, C, and D, bought a hive of bees for £1; A paid one third, B one fourth, C one fifth, and D one sixth. How much did each pay of the 20s.?

Let x = the sum A paid in shillings;

then $\frac{1}{3} : \frac{1}{4} :: x : \frac{3x}{4}$ = the sum B paid in shillings,

$\frac{1}{3} : \frac{1}{5} :: x : \frac{3x}{5}$ = the sum C paid in shillings,

$\frac{1}{3} : \frac{1}{6} :: x : \frac{x}{2}$ = the sum D paid in shillings;

$$\therefore x + \frac{3x}{4} + \frac{3x}{5} + \frac{x}{2} = 20;$$

clearing the equation of fractions,

$$20x + 15x + 12x + 10x = 400, \text{ or } 57x = 400;$$

$$\therefore x = \frac{400}{57} = 7\tfrac{1}{57}\text{s. A paid;}$$

whence, A $7\frac{1}{57}$s., B $5\frac{15}{57}$s., C $4\frac{12}{57}$s., and D $3\frac{20}{57}$s.

18. A person bought a quantity of eggs at two a penny, and as many at three a penny, and sold them all out at five for two-pence, and by so doing lost 4d. What number of eggs did he buy?

Let x = the number of eggs at each price,

or $2x$ = the eggs bought;

then $\frac{x}{2}$ = the price of the first quantity in pence,

and $\frac{x}{3}$ = the price of the 2nd quantity in pence;

$$\therefore \frac{x}{2} + \frac{x}{3} = \text{the cost price;}$$

and $5 : 2 :: 2x : \frac{4x}{5}$ = the selling price, which, by the question, is less by 4d. than the cost price,

$$\therefore \frac{x}{2}+\frac{x}{3}-\frac{4x}{5}=4;$$

clearing the equation of fractions,

$$15x+10x-24x=120; \therefore x=120.$$

18. A person at play lost 10s. more than one half of his money; he afterwards lost 10s. more than half the sum he had left: on going home, he was met by a robber, who took from him half of what he then had, and 10s. more; when he arrived at home, he found that he had only £5 left. What sum had he at first?

Let x = the sum he had at first in pounds;

then $\frac{x}{2}+\frac{1}{2}$ = the sum he lost at play;

$\therefore x-\frac{x}{2}-\frac{1}{2}$, or $\frac{x}{2}-\frac{1}{2}$, or $\frac{x-1}{2}$ = the sum he had left after his first loss;

Also, $\frac{x-1}{2}\div 2=\frac{x-1}{4}$, and $\frac{x-1}{4}+\frac{1}{2}$ or $\frac{x+1}{4}$ = the sum he lost the second time;

$\therefore \frac{x-1}{2}-\frac{x+1}{4}=\frac{x-3}{4}$ = the sum he had left after his second loss;

Also, $\frac{x-3}{4}\div 2=\frac{x-3}{8}$, and $\frac{x-3}{8}+\frac{1}{2}$ = the sum he was robbed of,

then $\frac{x-3}{4}-\frac{x-3}{8}-\frac{1}{2}=5$, by the question;

clearing this equation of fractions,

$$2x-6-x+3+4=40;$$

$\therefore$ by transposition, $x=47$.

the gin 15s. per gallon; and he paid £4. 10s. more for the brandy than for the gin. Required the content of each cask in gallons.

Ans. 20 gallons of brandy, and 34 of gin.

14. The perimeter of a triangle is 35 chains; the base is 7 chains longer than one side, and 3 chains longer than the other. Required the length of each side.

Ans. 15, 12, and 8.

15. A contributed 15s. more than B towards the relief of the poor, and if 25s. be added to three quarters of B's money, the sum will be equal to A's money. How much did each contribute?

Ans. A 55s., and B 40s.

16. A merchant sold two pieces of cloth, by one of which he gained 15s. more than by the other, and his whole gain was £2. 5s. less than four times the less gain. How much did he gain by each piece?

Ans. 45s. and 30s.

17. Ten years ago, A's age was in proportion to B's as 2 to 3, but they now find that the proportion is as 3 to 4. Required each person's age.

Ans. A's 30, and B's 40.

18. A cistern, which holds 945 gallons, is filled in half an hour by 3 pipes; the second conveys four-fifths as many gallons as the first, and the third three-fourths as many gallons as the first and second together, per minute. How much flows through each pipe per minute?

Ans. 10, 8, and 13½ gallons.

19. Divide the number 44 into two such parts, that the greater increased by 5, and the less by 7, their sums will be in the proportion of 4 to 3.

Ans. 27 and 17.

20. A farmer hired a labourer for 40 days, at 3s. 4d. per day, for every day he worked; but for every day he played he was to forfeit 16d.: at the end of the time he received £3. 3s. 4d. How many days did he work, and how many did he play?

Ans. He worked 25 days, and played 15.

21. Two merchants, A and B, entered into partnership with a joint stock of £500, by which they gained £160; of this A received £32 more than B. Required each person's stock.

Ans. A's stock £300, and B's £200.

22. A person bought a certain number of oranges at five for two pence, and afterwards sold one half of them at two for a penny, and the remainder at three a penny, and by so doing gained four pence. How many oranges were bought?

Ans. 240.

23. What two numbers are in the proportion of 3 to 4, and whose sum is to the sum of their squares as 7 to 50?

Ans. 6 and 8.

24. A draper sold a certain quantity of cloth for £33. 7s. 6d.; for one third of the whole he received 9s. 6d. per yard, for one fourth of the whole he received 11s. per yard, and for the remainder he received 12s. 6d. per yard. How many yards did he sell?

Ans. 60 yards.

25. A certain sum of money put out to interest amounted to £297. 12s. in eight months, and to £306 in fifteen months. Required the sum and rate per cent.

Ans. £288, at 5 per cent.

26. Four merchants, A, B, C, and D, entered into partnership, by which they gained a certain sum; of

which, A had £800 less than one half of the whole, B £120 more than one fourth of the whole, C half as much as A, and D only two thirds as much as B. How much had each?

Ans. A £2200, B £1620, C £1100, and D £1080.

27. A charitable person gave twenty-four shillings amongst some poor men, women, and children; each man received twelve pence, each woman nine pence, and each child three pence. Now, if there had been two women more, there would have been twice as many women as men; and had there been four children more, their number would have been equal to three times the number of women. Required the number of each.

Ans. 7 men, 12 women, and 32 children.

28. Two persons, A and B, received each five shillings per day for their labour; A saved six pence per day more than B, and at the end of 50 days, the amount of his savings was double B's, and two day's expenses over. What did each person spend per day?

Ans. A spent 4s. 2d., and B 4s. 8d.

29. A mercer bought a quantity of silk at 3s. 2d. per yard, and retailed one third of it at 4s. per yard, one quarter of the whole at 3s. 8d. per yard, one fifth of the whole quantity at 3s. 6d. per yard, and the remainder at 3s. 4d. per yard, and by so doing gained 15s. 2d. How many yards did he buy?

Ans. 30.

30. A and B began to play with equal sums of money; A won 20 sovereigns; B afterwards won half the money A had; he then found that he had twice as much money as A. How much money had each person at first?

Ans. 60 sovereigns.

31. A person being asked the hour of the day, answered, that if three eights of the hours remaining till midnight be multiplied by 4, the product will as much exceed 12 hours, as half the present time from noon is below 4. Required the hour of the day.

Ans. 2 o'clock, P. M.

32. A bill of £36. 5s. was paid in half guineas and crowns, and the number of pieces used of both sorts was 90. How many were there of each?

Ans. 50 half guineas, and 40 crowns.

33. Two persons, A and B, have each the same annual income; A saves one fifth of his yearly, and B, by spending £60 per annum more than A, at the end of three years finds himself £100 in debt. Required each person's income and expenditure.

Ans. £133. 6s. 8d. yearly income; A spends £106. 13s. 4d., and B £166. 13s. 4d.

34. Divide 36 into three such parts, that one half of the first, one third of the second, and one fourth of the third may be equal to each other.

Ans. 8, 12, and 16.

35. What number is that, which being divided into three equal parts, and also into four equal parts, whose continued product, in each case, is the same?

Ans. $9\frac{13}{27}$.

36. After A had been dispatched 16 hours on his journey from Leeds to London, a distance of 192 miles, B started to overtake him; in order to do this, he found that his rate of travelling must be to A's as 3 to 2. How many miles did each person travel per hour?

Ans. A 4 miles, and B 6 miles.

37. A draper bought a quantity of cloth for £62 10s., after parting with 10 yards to a friend, he sold

the remainder for £55, by which he gained 10 per cent. How many yards did he buy, and what did it cost him per yard?

Ans. 50 yards at 25s. per yard.

38. Bought a horse for a certain price, and found that if I could sell him for 60 guineas, I should gain twice as much per cent. as I should if I sold him for 50 guineas. Required his prime cost.

Ans. 40 guineas.

39. A merry young fellow in a short time spent one-fifth of his fortune; by the advice of his friends he gave £2200 for a place in the Guards; his profusion continued till he had only 880 guineas left, which he found was three-twentieths of his money after the commission was bought. Pray what was his fortune at first?

Ans. £10450.

40. Three boys met a girl carrying apples to the market: the first took half what she had, but returned to her ten; the second took one-third, but returned two; and the third took away half those she had left, but returned one: she had then twelve apples left. How many had she at first?

Ans. 40.

41. By sliding out the bottom of a ladder, which was placed upright against a wall, 10 feet from the wall, I observed the top to fall one yard. Required the length of the ladder.

Ans. 18 feet 2 inches.

42. A mason having to form a stone for a circular window top, to rise 20 inches, what radius must he take to strike the arc, the breadth of the window being 4 feet 2 inches?

Ans. 25.625 inches.

43. Required the area of that square, the paving of which, at two shillings per square yard, cost as much as inclosing it at six shillings per yard, lineal measure.

Ans. 144.

44. Required the side of that cube, whose solidity contains as many cubic inches, as the sides do square inches.

Ans. 6.

45. Required the side of that cube, whose superficies is to its solidity as 6 to 11.

Ans. 11.

46. Required the diameter of that globe, whose solidity and superficial content are equal to each other.

Ans. 6.

47. Required the diameter of that globe, whose superficies is to its solidity as 2 to 3.

Ans. 9.

48. Required the side of that equilateral triangle, whose area cost as much paving, at eight pence per foot, as pallisading the three sides did at a guinea per yard.

Ans. 72.746.

SIMPLE EQUATIONS

OF TWO OR MORE UNKNOWN QUANTITIES.

In the solution of equations containing two or more unknown quantities, there must be as many independent equations as there are unknown quantities, in order that the unknown quantities may have determinate values, which may be found by the following rules:—

Rule I.

1. *When the given question contains two equations, each containing two unknown quantities, as in the first example, find the value of the same unknown quantity, in terms of the rest of the equation, in each equation, then, by the 5th axiom in Def. 4, things which are equal to the same things are equal to each other, a new equation will be formed containing one unknown quantity only, which may be solved by the rules already given.*

2. *But when the given question contains three equations, with three unknown quantities, as in example 2d. find the value of the same unknown quantity, in each equation, in terms of the rest, as in example 2d., then, by axiom 5, two new equations may be formed, each containing two unknown quantities, which may be solved by the first part of the rule.*

3. *Also, when the given question contains four equations, each containing four unknown quantities, find the value of the same unknown quantity, in terms of the rest of the equation, then, as above, three new equations may be formed, each containing three unknown quantities; therefore, by the second part of the rule, the value of each may be found, and so on, for any other number of equations.*

1. Given $3x+2y=18$, and $2x-y=5$, } to find the values of x and y.

Here, from the first equation, $3x=18-2y$;

$$\therefore\ x=\frac{18-2y}{3}.\ (A)$$

From the second equation, $2x=5+y$;

$$\therefore\ x=\frac{5+y}{2}.\ (B)$$

Then (by ax. 5th.), $\frac{5+y}{2}=\frac{18-2y}{3}$, a new equation, containing one unknown quantity only.

Multiplying this equation by 6, the least common multiple of 2 and 3,

$$15+3y=36-4y;$$

by transposition, $3y+4y=36-15$,

or $7y=21$; $\therefore\ y=3$.

Substituting this value of y, which is 3, in the equation (A), $x=\frac{18-2\times3}{3}=\frac{12}{3}=4$;

Or, if the value of y be substituted in the equation (B)

$$x=\frac{5+3}{2}=\frac{8}{2}=4.$$

whence, $x=4$, and $y=3$.

Or thus:—

From the first equation, $2y=18-3x$;

$$\therefore\ y=\frac{18-3x}{2}.\ (A)$$

NOTE.—When there are more unknown quantities than independent equations, some of these quantities cannot be determined, except in terms of the rest; but if there be more independent congruous equations than unknown quantities, some of the equations are unnecessary.

Also, from the second equation, $y=2x-5$; (B)

$\therefore$ (by ax. 5th.), $2x-5=\frac{18-3x}{2}$;

multiplying by 2, $4x-10=18-3x$;

by transposition, $4x+3x=18+10$,

or $7x=28$; $\therefore x=4$;

substituting this value of x, in the equation (A),

$$y=\frac{18-3\times4}{2}=\frac{6}{2}=3\text{ ;}$$

or if this value of x be substituted in the equation (B),

$$y=2\times4-5=8-5=3\text{ ;}$$

whence, $x=4$, and $y=3$.

2. Given $\left.\begin{array}{l}3x+2y+4z=28,\\ 6x-3y+2z=11,\\ \text{and } 9x+6y-8z=4,\end{array}\right\}$ to find the values of x, y, and z.

Here, from the first equation, $4z=28-3x-2y$;

$$\therefore z=\frac{28-3x-2y}{4}\text{ ; (A)}$$

from the second equation, $2z=11-6x+3y$;

$$\therefore z=\frac{11-6x+3y}{2}\text{ ; (B)}$$

also, from the third equation, $8z=9x+6y-4$;

$$\therefore z=\frac{9x+6y-4}{8}\text{ ; (C)}$$

then (by ax. 5th.), $\frac{28-3x-2y}{4}=\frac{11-6x+3y}{2}$;

multiplying this new equation by 4,

$$28-3x-2y=22-12x+6y\text{ ;}$$

by transposition, $28-22=3x-12x+2y+6y$,

or $8y-9x=6$;

$$\therefore y=\frac{6+9x}{8}\text{ ; (D)}$$

also (by ax. 5), $\frac{11-6x+3y}{2}=\frac{9x+6y-4}{8}$;

multiplying this new equation by 8,

$$44-24x+12y=9x+6y-4;$$

by transposition, $44+4=24x+9x+6y-12y$,

or $33x-6y=48$;

$$\therefore y=\frac{33x-48}{6}; \text{ (E)}$$

$$\therefore \text{(by ax. 5th)}, \frac{33x-48}{6}=\frac{9x+6}{8};$$

multiplying this equation by 24,

$$132x-192=27x+18;$$

by transposition, $132x-27x=192+18$,

or $105x=210$; $\therefore x=2$.

Substituting this value of x, in the equation E,

$$y=\frac{\overline{33\times2}-48}{6}=\frac{18}{6}=3.$$

Also, substituting the values of x and y, in the equation A,

$$z=\frac{28-\overline{3\times2}-\overline{2\times3}}{4}=\frac{28-6-6}{4}=\frac{16}{4}=4,$$

whence, $x=2$, $y=3$, and $z=4$.

NOTE.—The value of z may be found by substituting the values of x and y, in any of the equations, A, B, or C. Also, the value of y may be found by substituting the value of x, in the equations D or E.

RULE II.

Multiply or divide the given equations by any number that may be convenient, to make one unknown quantity in each equation the same; then, if these quantities have like signs, the difference of the equations must be taken, in order to exterminate one of the unknown quantities.

But if the equal quantities have unlike signs, then their sum must be taken, in order to exterminate one of the unknown quantities.

1. Given $3x+2y=18$, and $2x-y=5$, } to find the values of x and y.

Here, multiplying the second equation by 2,

$$4x-2y=10;$$

but $3x+2y=18$, (the first equation,)

by addition, $7x=28$*; $\therefore x=4$.

Substituting this value of x, in any of the above equations, y will be found $=3$.

Or thus:—

Multiplying the first equation by 2, and the second by 3,

$$6x+4y=36; \quad (A)$$

and $6x-3y=15$;

by subtraction, $7y=21$†; $\therefore y=3$.

Substituting this value of y, in the equation (A),

$$6x+\overline{4\times 3}=36,$$

or, $6x+12=36$;

by transposition, $6x=36-12=24$;

$$\therefore x=\frac{24}{6}=4.$$

Or thus:—

Dividing the first equation by 2,

$$\frac{3x}{2}+y=9;$$

* Note.—The $-2y$ and $2y$ are like quantities, and have unlike signs; therefore, the sum of the equations is taken in order to exterminate the y.

† Here the unknown quantity x is exterminated.

but $2x-y=5$;

by addition, $\frac{7x}{2}=14$;

$\therefore 7x=28$, or $x=4$.

Substituting this value of x, in one of the above equations, $y=3$.

2. Given $3x+2y+4z=28$, $6x-3y+2z=11$, and $9x+6y-8z=4$, } to find the values of x, y, and z.

Here, multiplying the second equation by 2,

$12x-6y+4z=22$;

but $3x+2y+4z=28$; (the 1st equation)

by subtraction, $-9x+8y=6$. (A)

NOTE.—By this operation, z is exterminated.

Also, multiplying the first equation by 2, and adding the product to the third equation,

$6x+4y+8z=56$, (twice the 1st equation,)

but $9x+6y-8z=4$, (the third equation,)

by addition, $15x+10y=60$. (B)

By this operation the z is also exterminated, therefore, here are two equations, viz. A and B, each containing two unknown quantities.

Multiplying the equation A by 10, and the equation B by 8,

$-90x+80y=60$ (10 times the equation A),

$120x+80y=480$ (8 times the equation B),

$210x=420$, by subtraction; $\therefore x=2$.

Substituting this value of x, in the equation B,

$15\times2+10y=60$, or $30+10y=60$;

by transposition, $10y=60-30=30$;

$$\therefore\ y=\frac{30}{10}=3.$$

Also, substituting the values of x and y in the first equation, viz. $3x+2y+4z=28$, it becomes

$$6+6+4z=28;$$

by transposition, $4z=28-6-6$,

$$\text{or } 4z=16;\ \therefore\ z=\frac{16}{4}=4.$$

Rule III.

Find the value of one of the unknown quantities, in terms of the rest of the equation, and substitute its value, thus found, in the other equation.

1. Given $3x+2y=18$, and $2x-y=5$, } to find the values of x and y.

From the first equation, by Rule I, $x=\frac{18-2y}{3}$. (A)

Substituting this value of x in the second equation,

$$2.\left(\frac{18-2y}{3}\right)-y=5,\ \text{or}\ \frac{36-4y}{3}-y=5;$$

multiplying this equation by 3,

$$36-4y-3y=15;$$

by transposition, $36-15=4y+3y$,

$$\text{or } 21=7y;\ \therefore\ y=3.$$

Substituting this value of y, in the equation (A),

$$x=\frac{18-6}{3}=\frac{12}{3}=4.$$

Or thus:—

From the second equation, $y=2x-5$, (A)

substituting this value of y, in the first equation,

$3x+2.(2x-5)=18$, or $3x+4x-10=18$;

by transposition, $3x+4x=18+10$, or $7x=28$;

$$\therefore\ x=4.$$

Substituting this value of x, in the equation (A),

$$y=3.$$

NOTE.—This method of solution is applicable to the second example, or to any question containing four or more equations, with the same number of unknown quantities.

EXAMPLES,

With their solutions, by the 2nd. Rule.

1. Given $\frac{x}{3}+\frac{y}{4}=8$, and $\frac{2x}{8}+\frac{3y}{12}=7$, } to find the values of x and y.

Clearing the first equation of fractions,

$$4x+3y=96;\ (A)$$

clearing the second equation of fractions,

$$6x+6y=168;$$

dividing this equation by 2,

$$3x+3y=84;$$

but $4x+3y=96$;

by subtraction, $x=12$;

whence, by transposing the equation (A), $3y=84-3x$;

substituting this value of x, in this equation,

$$3y=84-36=48;\ \therefore\ y=16.$$

2. Given $5x-20=3y+2$, and $3x+6=7y-12$, } to find the values of x and y.

From the first equation, by transposition,

$$5x-3y=22;\ (A)$$

from the second equation, by transposition,

$$3x-7y=-18;$$

multiplying the first of these equations by 7, and the second by 3,

$$35x-21y=\ 154;$$
$$9x-21y=-54;$$

by subtraction, $26x\ \ =\ 208;\ \therefore\ x=8,$

whence, by transposing the equation (A), and substituting the value of x therein,

$$3y=5x-22=40-22=18;\ \therefore\ y=6.$$

3. Given $\frac{2x-y}{5}+7\frac{1}{2}=\frac{x+2y+2}{4}+6,$ and $12-\frac{x+y}{4}=\frac{x+3y}{6}+4,$ } to find the values of x and y.

Multiplying the first equation by 20,

$$8x-4y+150=5x+10y+10+120;$$

by transposition, $8x-5x-4y-10y=10+120-150,$

$$\text{or } 3x-14y=-20.$$

Multiplying the second equation by 12,

$$144-3x-3y=2x+6y+48;$$

by transposition, $-3x-2x-3y-6y=48-144,$

$$\text{or } 5x+9y=96;$$

multiplying this equation by 3, and the other by 5,

$$15x+27y=\ 288;$$
$$\text{and } 15x-70y=-100;$$

by subtraction, $97y=\ 388;\ \therefore\ y=4;$

whence, by transposition and substitution,

$$5x=96-9y=96-36=60;\ \therefore\ x=12.$$

4. Given $\frac{ax+by}{4}=3b,$ and $\frac{2ax-by}{3}=c,$ } to find the values of x and y.

Multiplying the first equation by 4, and the second by 3,

$$ax+by=12b\text{; (A)}$$
$$\text{and } 2ax-by=3c\text{;}$$

by addition, $3ax=12b+3c$;

$$\therefore\ x=\frac{12b+3c}{3a}=\frac{4b+c}{a}\text{;}$$

substituting this value of x, in the equation (A),

$$4b+c+by=12b\text{;}$$

by transposition, $by=12b-4b-c=8b-c$;

$$\therefore\ y=\frac{8b-c}{b}.$$

5. Given $2x-\dfrac{4y-\frac{x+4}{6}}{4}=12+\dfrac{x+\frac{2y-1}{2}}{5}$, and $2x : 8y :: 4 : 5$, } to find the values of x and y.

Multiplying the first equation by 6, the least common multiple of 6 and 2,

$$12x-\frac{24y-x-4}{4}=72+\frac{6x+6y-3}{5}.$$

Also, multiplying this equation by 20,

$$240x-120y+5x+20=1440+24x+24y-12\text{;}$$

by transposition, $240x+5x-24x-120y-24y=1440-12-20$, or $221x-144y=1408$. (A)

From the second equation, Theorem 2,

$$10x=32y,\ \therefore\ x=\frac{32y}{10}.\ \text{(B)}$$

Substituting this value of x, in the equation (A), it becomes, $\frac{7072y}{10}-144y=1408$;

$$\therefore\ 7072y-1440y=14080,\text{ or } 5632y=14080\text{;}$$
$$\therefore\ y=2\tfrac{1}{2}.$$

Substituting this value of y, in the equation (B),

$$x=\frac{32\times 2\frac{1}{2}}{10}=\frac{80}{10}=8.$$

EXAMPLES FOR PRACTICE,

In Simple Equations of two unknown Quantities.

1. Given $4x+6y=36$, and $2x+5y=26$, } to find the values of x and y.

Ans. $\begin{cases} x=3. \\ y=4. \end{cases}$

2. Given $4x-y=17$, and $5x+3y=51$, } to find the values of x and y.

Ans. $\begin{cases} x=6. \\ y=7. \end{cases}$

3. Given $5x+4y=22$, and $7x-5=3y$, } to find the values of x and y.

Ans. $\begin{cases} x=2. \\ y=3. \end{cases}$

4. Given $4x-y+4=21$, and $5x+3y-9=42$, } to find the values of x and y.

Ans. $\begin{cases} x=6. \\ y=7. \end{cases}$

5. Given $\frac{x}{4}+\frac{y}{3}=6$, and $\frac{x}{3}+\frac{y}{4}=6\frac{1}{4}$, } to find the values of x and y.

Ans. $\begin{cases} x=12. \\ y=9. \end{cases}$

6. Given $7x+\frac{2y}{5}=30$, and $7y+\frac{2x}{5}=36\frac{3}{5}$, } to find the values of x and y.

Ans. $\begin{cases} x=4. \\ y=5. \end{cases}$

7. Given $\frac{3x+y}{5}+x=4$, and $\frac{7x-4}{2}=2y-3$, } to find the values of x and y.

Ans. $\begin{cases} x=2. \\ y=4. \end{cases}$

8. Given $\frac{4x+y}{2}-\frac{2x+2y-4}{4}=y-2$, and $\frac{3y+4}{7}+\frac{4}{7}=x+1$, } to find the values of x and y.

Ans. $\begin{cases} x=4. \\ y=9. \end{cases}$

9. Given $\frac{x+y}{4}+6=12-y$, and $\frac{x+y}{2}+13=x+y+4$, } to find the values of x and y.

Ans. $\begin{cases} x=9. \\ y=3. \end{cases}$

10. Given $\frac{3x+2y}{11}=\frac{3x+6}{9}$, and $\frac{2x+3y+4}{9}=3$, } to find the values of x and y.

Ans. $\begin{cases} x=4. \\ y=5. \end{cases}$

11. Given $\frac{3x+3}{7}-\frac{3x+2y-1}{9}+5=\frac{4x-y+11}{6}$, and $\frac{2x+6}{3}=\frac{x+y+1}{2}$, } to find the values of x and y.

Ans. $\begin{cases} x=6. \\ y=5. \end{cases}$

12. Given $7+\frac{x+y}{3}=\frac{4x+y}{3}+y-2x$, and $x+1 : y :: 6 : 14$, } to find the values of x and y.

Ans. $\begin{cases} x=3\frac{1}{2}. \\ y=10\frac{1}{2}. \end{cases}$

13. Given $12x - \frac{3y-4x+26}{5} - 10y = 4y + \frac{x+2y+1}{2}$, and $3x+y+1 : 2x+4 :: 6 : 5$, to find the values of x and y.

Ans. $\begin{cases} x=3. \\ y=2. \end{cases}$

14. Given $\frac{3x+5y+10}{3} - \frac{2x+2y}{9} = \frac{22x+85}{15}$, and $\frac{3x+2y+1}{8} : \frac{3x-2y+5}{3} :: 9 : 12$, to find the values of x and y.

Ans. $\begin{cases} x=5. \\ y=4. \end{cases}$

15. Given $\frac{4x-2y}{3} + 6\frac{2}{3} = 12 - \frac{x+y}{4} - \frac{1}{2}$ and $\frac{2x+3y}{6} + 7\frac{3}{4} = 11\frac{1}{6} + \frac{x+y}{12}$, to find the values of x and y.

Ans. $\begin{cases} x=4\frac{1}{2}. \\ y=5\frac{1}{2}. \end{cases}$

16. Given $\frac{6x-2y}{4} + 9\frac{1}{2} = \frac{6y+4x}{2} - \frac{3y-x}{2}$, and $\frac{y^2+x+8}{y} = y+3$, to find the values of x and y.

Ans. $\begin{cases} x=2\frac{1}{2}. \\ y=3\frac{1}{2}. \end{cases}$

17. Given $3x + \frac{4x+3}{2} = y+21 - \frac{4x+38}{8}$ and $x + \frac{2x+4y}{5} = 12 - x$, to find the values of x and y.

Ans. $\begin{cases} x=3\frac{1}{2}. \\ y=4\frac{1}{2}. \end{cases}$

18. Given $\frac{28}{x} = \frac{14}{y-3}$, and $\frac{x+y}{3} - \frac{3x-2y}{2} + \frac{3x}{6} = 8\frac{1}{3} - \frac{3x+1}{3}$, to find the values of x and y.

Ans. $\begin{cases} x=4. \\ y=5. \end{cases}$

19. Given $\dfrac{\frac{3x+y}{4}}{3}+\dfrac{9+2y}{7}=x+\dfrac{1}{2}$, and $63x+y=258$, } to find the values of x and y.

Ans. $\begin{cases} x=4. \\ y=6. \end{cases}$

20. Given $\dfrac{4xy-y}{2x}=2y-1$, and $7x-\dfrac{12y+1}{7}=9-\dfrac{4x+y}{6}$, } to find the values of x and y.

Ans. $\begin{cases} x=2. \\ y=4. \end{cases}$

21. Given $ax-\dfrac{b-y}{4}=3ax$, and $a:b::x:y$, } to find the values of x and y.

Ans. $\begin{cases} x=\dfrac{ab}{b-8a^2}. \\ y=\dfrac{b^2}{b-8a^2}. \end{cases}$

22. Given $5x-\dfrac{\frac{2y+4}{3}}{4}-13=\dfrac{5y}{4}+\dfrac{6x-\frac{5y-6}{7}}{17}+9$, and $\dfrac{2x+y}{8}:\dfrac{x+y}{2}::2:5$, } to find the values of x and y.

Ans. $\begin{cases} x=6. \\ y=4. \end{cases}$

23. Given $8x+5y+6z=56$, $3x-2y+3z=7$, and $2x+6y-2z=26$, } to find the values of x, y, and z.

Ans. $\begin{cases} x=3. \\ y=4. \\ z=2. \end{cases}$

24. Given $2w+3x-2y+z=15$, $4x-y+3w-2z=9$, $3z+4y-2x-w=17$, and $4y-5x+2z+4w=10$, } to find the values of w, x, y, and z.

Ans. $w=2$, $x=4$, $y=3$, and $z=5$.

THE APPLICATION

Of Simple Equations of two unknown quantities, with their solutions.

1. Says A to B, if six years were added to my age, I should be as old as you; but if eight years were added to your age, you would be twice as old as me. Required each person's age.

Let x= A's age in years,
and y= B's age in years;
Then $x+6=y$; and $y+8=2x$;
transposing the first Equation, $y-x=6$;
transposing the second Equation, $-y+2x=8$;

$\therefore$ by Addition, $x=14$.

hence, $y=x+6=14+6=20$;

$\therefore$ A is 14, and B 20 years of age.

2. Says A to B, if I had four of your marbles, I should have twice as many as you; but says B, if I had six of your marbles I should have eight times as many as you. Required each person's number.

Let x=the number A had,
and y=the number B had;
then $x+4$=the number A had after receiving four of B;
and $y-4$=the number B had after giving four to A;
$\therefore x+4=2.(y-4)$, or $x+4=2y-8$;
by transposition, $2y-x=12$;
but $y+6$=what B had after receiving six from A,
and $x-6$=what A had after giving six to B;
$\therefore y+6=8.(x-6)$, or $y+6=8x-48$;
$\therefore$ by transposition, $8x-y=54$.

Multiplying this equation by 2,

$$16x-2y=108;$$

but $2y-x=12;$

by Addition, $15x=120;\ \therefore x=8;$

hence, $2y=x+12=8+12=20;\ \therefore y=10,$

$\therefore$ A had 8, and B 10.

3. A draper sold a certain quantity of cloth, consisting of blue and black, for £59. For the blue he charged 18s. and for the black 16s. per yard. He found, however, that if he had received 16s. for the blue, and 18s. for the black, per yard, he should have received £60. How many yards did he sell of each sort?

Let x=the number of yards of blue,

and y=the number of yards of black;

$\therefore 18x$=value of the blue,

and $16y$=value of the black;

but if, by the question, the blue was charged 16s. and the black 18s. per yard,

then $16x$=value of the blue,

and $18y$=value of the black;

then $18x+16y=1180,$

and $16x+18y=1200;$

by Subtraction, $-2x+2y=20;$

multiplying this equation by 9,

$$-18x+18y=180;$$

but $16x+18y=1200;$

by Addition, $34x=1020;\ \therefore x=30;$

hence, $2y=20+2x=20+60=80;\ \therefore y=40;$

$\therefore$ he sold 30 yards of blue, and 40 of black.

4. Bought a certain quantity of sugar, at 45s. per cwt., and a quantity of currants at 58s. per cwt., for £88. 10s., and sold the sugar at 2s. and the currants at 3s. per cwt. advance, and gained thereby £4. 5s. How many cwt. of each did he buy?

Let x=the number of cwts. of sugar,
and y=the number of cwts. of currants;
then $45x$=value of the sugar in shillings,
and $58y$=value of the currants in shillings;
$\therefore 45x+58y=1770$, (shillings in £88. 10s.)
also, $2x$=the shillings gained by the sugar,
and $3y$=the shillings gained by the currants;
$\therefore 2x+3y=85$.

Multiplying this equation by 58, and the other by 3,

$$116x+174y=4930,$$
$$135x+174y=5310;$$

by Subtraction, $19x = 380$; $\therefore x=20$;

hence, $3y=85-2x=85-40=45$; $\therefore y=15$;

whence, he bought 20 cwt. of sugar, and 15 cwt. of currants.

5. What fraction is that, to the numerator of which if one be added, its value becomes one-half; but if three be added to the denominator, its value will be one-third?

Let $\frac{x}{y}$=the fraction required;

then $\frac{x+1}{y}=\frac{1}{2}$; $\therefore 2x+2=y$;

and $\frac{x}{y+3}=\frac{1}{3}$; $\therefore 3x=y+3$;

by Subtraction, $x-2=3$;

$\therefore x=5$, the numerator,

hence, $y=2x+2=10+2=12$, the denominator;
whence, the fraction is $\frac{5}{12}$.

6. A certain sum of money put out to interest for ten months, at a certain rate per cent. amounted to £361. 13s. 4d.; and in eighteen months it amounted to £371. Required the sum and rate per cent.

Let x=the sum in pounds,
and y=the rate per cent.;

then $\frac{xy}{100}$=the interest for one year;

and $\overset{m}{12} : \frac{xy}{100} \overset{£.}{::} \overset{m}{10} : \frac{xy}{120} \overset{£.}{=}$ the interest for ten months;

$$\therefore x+\frac{xy}{120}=361\tfrac{1}{3};$$

multiplying this equation by 120,

$$120x+xy=43400; \quad \text{(A)}$$

also, $\overset{m}{12} : \frac{xy}{100} \overset{£.}{::} \overset{m}{18} : \frac{18xy}{1200}=\frac{3xy}{200}=$ the interest for eighteen months.

$$\therefore x+\frac{3xy}{200}=371;$$

multiplying this equation by 200,

$$200x+3xy=74200;$$

but $360x+3xy=130200$ (= 3 times equation A)

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by Subtraction, $160x = 56000$; $\therefore x=350$;
substituting this value of x in the equation A,

$$42000+350y=43400;$$

by transposition, $350y= 1400$; $\therefore y=4$;
whence, the sum put out is £350, at four per cent. per annum.

7. There is a number, consisting of two digits, the first of which is greater than the second, and if the number be divided by one more than the sum of the digits, the quotient will be 6; but if the digits be inverted, and that number be divided by the sum of the digits, the quotient will be 4. Required the number.

Let x denote the first digit, or that in the ten's place, and y denote the 2nd. digit, or that in the unit's place;

then $x+y$=the sum of the digits,

and $10x+y$=the number;

$\therefore$ by the question, $\frac{10x+y}{x+y+1}=6$,

or $10x+y=6x+6y+6$;

by transposition, $4x-5y=6$.

Also, by the question, $\frac{10y+x}{x+y}=4$,

or $10y+x=4x+4y$;

by transposition, $6y-3x=0$, or $2y-x=0$;

Multiplying this equation by 4,

$$8y-4x=0;$$

but $4x-5y=6$;

by addition, $3y = 6$,

or $y = 2$; $\therefore$ $x=2y=4$;

whence, the number is 42.

NOTE. Inverting the digits, is placing the y, which is in the unit's place, in the ten's place, and placing the x, which is in the ten's place, in the unit's place. Thus, in the above example, $10x+y$ denotes the number; but when the digits are inverted, $10y+x$ denotes the number.

EXAMPLES FOR PRACTICE

In Simple Equations of two unknown quantities.

1. A draper bought two pieces of cloth, one at 18s. per yard, and the other at 16s. per yard, for £43; and three times the length of the piece at 18s. per yard, exceeded four times the length of the piece at 16s. per yard by ten yards. Required the length of each piece.

Ans. 30 yards at 18s., and 20 yards at 16s.

2. A bill of £27. 5s. was discounted in half guineas and half crowns, and three times the number of half guineas exceeded twice the number of half crowns by 20. Required the number of each.

Ans. 40 half guineas, and 50 half crowns.

3. A mercer bought two pieces of silk for £8. 12s. the first 18 yards and the second 20 yards, and 7 yards of the first piece cost 3s. more than 5 yards of the second piece. Required the price per yard of each piece.

Ans. First 4s. and second 5s. per yard.

4. A gentleman bought a quantity of brandy at 5s. a bottle, and a quantity of rum at 3s. 9d. a bottle, and paid twice as much for the brandy as for the rum. Had he bought as many bottles of brandy as he bought of rum, and as many bottles of rum as he bought of brandy, he would only have paid 5s. more for the rum than for the brandy. How many bottles of each did he buy?

Ans. 12 of brandy, and 8 of rum.

5. A farmer sold 30 bushels of wheat and 40 bushels of barley, and received for the whole £25: he also sold 50 bushels of wheat and 30 bushels of barley, and received for this bargain £7. 10s. more than he did the former time. Required the price of each per bushel.

Ans. Wheat 10s., and barley 5s.

6. A certain sum of money put out to interest, amounted to £297. 12s. in eight months, and in fifteen months to £306. Required the sum and rate per cent.

Ans. £288 at 5 per cent.

7. Required that fraction, whose numerator increased by 2, its value becomes one-half, but if one be added to its denominator, its value becomes one-third.

Ans. $\frac{5}{14}$.

8. A butcher bought 12 sheep and 7 oxen for £150; he afterwards bought 21 sheep and 6 oxen for the same sum. Required the price of each.

Ans. Sheep £2 each, and the oxen £18 each.

9. A gentleman left his son and daughter each a legacy, part of which was to be paid at six months, and the remainder at twelve months after the father's death; at six months the executors paid them £800, giving to the son five-sixths of his legacy, and to the daughter three-fourths of her legacy: on winding up the affairs, at the end of twelve months, they found that there only remained for the children £100, of this they gave to the son three-fifths of what remained due to him, and to the daughter two-fifths of what remained due to her. How much did the father leave to each?

Ans. £600 to the son, and £400 to the daughter.

10. There is a number, consisting of two digits, the first of which is greater than the second; now, if the number be divided by twice the sum of the digits +2, the quotient will be 3; and if the digits be inverted, and that number be divided by 7 times the difference of the digits +2, the quotient will also be 3. Required the number.

Ans. 96.

11. There is a certain number, if to the product of whose digits you add twice the right-hand digit, the result will be seven times that digit; but if to six times the sum of the digits you add 5, it will be the number required. What is that number?

Ans. 53.

12. A and B played at marbles, B won three-fourths of what A had, and then lost one-half of his stock; then he had only two-thirds as many marbles as A; but when A gave him 10, they had each the same number. How many had each at first?

Ans. A had 80 and B 20

13. Find two numbers, the greater of which shall be to the less, as their sum is to 21, and as their difference is to 3.

Ans. 16 and 12.

14. Find two numbers, whose sum, difference, and quotient, shall be as the numbers 4, 2, and 1, respectively.

Ans. 9 and 3.

15. A person measured a rectangular court yard, and found that if 5 feet were added to each of its dimensions, its area would be 250 feet more; but if 6 feet were added to the length, and 4 to the breadth, its area would be increased by 244 feet. Required its length, breadth, and area.

Ans. Length 25, breadth 20, and area 500.

16. A farmer bought calves at £2. 10s. each, and sheep at 30s. each, which together cost him £125, and found that the number of calves was to four-fifths of the number of sheep more than the number of calves, as 5 to 6. Required the number of each.

Ans. 20 calves, and 50 sheep.

17. After A had been dispatched 16 hours on his journey from Leeds to London, a distance of 192 miles, B started to overtake him as he entered the city; in order to do this, he found that the time it took him to travel 33 miles, added to the time it took A to travel 22, must be exactly 11 hours. How many miles did each travel per hour?

Ans. A 4 miles, and B 6 per hour.

18. A farmer mixed 28 bushels of barley at 2s. 4d. per bushel, with rye at 3s. per bushel, and wheat at 4s., so that the whole mixture may consist of 100 bushels, and be worth 3s. 4d. per bushel. How many bushels of rye and wheat did he mix with the barley?

Ans. 20 of rye, and 52 of wheat.

19. A person measured a rectangular field, and found if the length was increased by 4 chains and the breadth by 3, the adjacent sides would be in the ratio of 4 to 3; but if each side was decreased by 3, the ratio of the adjacent sides would be as 3 to 2. Required the area of the field.

Ans. 10*a*, 3*r*, 8*p*.

20. Find two numbers in the proportion of 4 to 5, from which two other numbers, in the proportion of 6 to 7, being respectively deducted from them, the remainders shall be in the proportion of 2 to 3, and their sum equal to 20.

Ans. 2 first numbers, 32 and 40, and the others, 24 and 28.

21. A draper sold two pieces of cloth, one 20 and the other 30 yards, for £45: he received for 8 yards of the shorter piece £2 more than he did for 4 yards of the other. Required the price of each piece per yard.

Ans. The shorter piece 15s., and the other 20s. per yard.

22. Required that number, one-fourth of which is equal to the difference of the squares of its digits; and if the number be added to three-fourths of itself, the digits will be inverted.

Ans. 12.

23. Required three numbers, such, that if one-half of the first, one-third of the second, and one quarter of the third be added together, their sum will be 62; also, if one-third of the first, one-fourth of the second, and one-fifth of the third be added together, their sum will be 47; but if one-fourth of the first, one-fifth of the second, and one-sixth of the third be added together, their sum will be 38.

Ans. The first 24, the second 60, and the third 120.

24. Three workmen, A, B, and C, being employed to build a house, it is found that A and B together can earn £2 in six days, A and C together can earn £2. 14s. in nine days, and B and C together can earn £4 in fifteen days. How much does each person receive per day?

Ans. A 3s. 8d., B 3s., and C 2s. 4d.

N. B.—Questions 23 and 24 are solved by three unknown quantities.

PURE QUADRATICS.

A *Pure Quadratic Equation* is that which contains the second power of the unknown quantity, without any other power of it.

Thus, $x^2=9$, $y^2=16$, $x^2=a$, &c. are pure quadratic equations, the value of the unknown quantity, in each case, being determined by extracting the square root of both sides of the equation, and they become as follows:—

$$x=\pm 3,\ y=\pm 4,\ x=\pm\sqrt{a}.$$

Here the sign of the value of the unknown quantity is either + or —.

Cor. The square root of any quantity being either + or —, pure quadratic equations admit of two answers. Thus, $+3\times+3$ make 9, and by Note 4th, Case I. in Multiplication, -3×-3 make 9, and each value, when substituted for x in the given equation, answering the conditions of the question.

If the square of the unknown quantity have a coefficient, or be connected with any known quantity by the sign + or —, such known quantity must be transposed to the other side of the equation, as in example first, then divide by the coefficient of the unknown quantity, and extract the square root of each side of the equation in order to find its value.

Pure Quadratics and others solved without completing the square.

1. Given $4x^2-16=48$, to find the values of x.

By transposition, $4x^2=48+16=64$, or $x^2=16$;

$$\therefore\ x=\pm 4.$$

2. Given $xy=48$, and $\frac{x}{y}=3$, } to find the values of x and y.

From the second equation, $x=3y$;

substituting this value of x in the first equation,

$$3y^2=48, \text{ or } y^2=16;$$

$$\therefore y=\pm 4, \text{ and } x=\pm 12.$$

3. Given $x-y : y :: 1 : 4$, and $x^2-y^2=81$, } to find the values of x and y.

By Theorem 3rd, from the first equation, $x : y :: 5 : 4$;

by Theorem 2nd, $4x=5y$; $\therefore y=\frac{4x}{5}$, or $y^2=\frac{16x^2}{25}$;

substituting this value of y^2 in the second equation,

$$x^2-\frac{16x^2}{25}=81;$$

$$\therefore 25x^2-16x^2=2025, \text{ or } x^2=225;$$

$$\therefore x=\pm 15, \text{ and } y=\pm 12.$$

4. Given $x+y : y :: 5 : 2$, and $x^2+y^2=117$, } to find the values of x and y.

By Theorem 4th, from the first equation,

$$x : y :: 3 : 2; \therefore y=\frac{2x}{3}, \text{ or } y^2=\frac{4x^2}{9};$$

substituting this value of y^2 in the second equation,

$$x^2+\frac{4x^2}{9}=117;$$

$$\therefore 9x^2+4x^2=1053, \text{ or } x^2=81;$$

$$\therefore x=\pm 9, \text{ and } y=\pm 6.$$

5. Given $x+y : x-y :: 7 : 1$, and $x^2+y^2=100$, } to find the values of x and y.

By Theorem 5th, from the first equation,

$$2x : 2y :: 8 : 6;$$

dividing the proportion by 2, (Theorems 9 and 10,)

$$x : y :: 4 : 3;$$

$$\therefore 4y=3x, \text{ or } y=\frac{3x}{4}, \text{ or } y^2=\frac{9x^2}{16};$$

substituting the value of y^2 in the second equation,

$$x^2+\frac{9x^2}{16}=100, \text{ or } 16x^2+9x^2=1600, \text{ or } x^2=64;$$

$$\therefore x=\pm 8, \text{ and } y=\frac{3x}{4}=\pm 6.$$

6. Given $x+y : x-y :: 4 : 2$, and $xy=48$, } to find the values of x and y.

By Theorem 6, $2y : 2x :: 2 : 6$;

dividing by 2, $y : x :: 1 : 3$; $\therefore x=3y$;

substituting this value of x in the second equation,

$$3y^2=48, \text{ or } y^2=16;$$

$$\therefore y=\pm 4, \text{ and } x=3y=\pm 12.$$

7. Given $x : x+y :: 2 : 5$, and $x^2+xy=40$, } to find the values of x and y.

From equation first, by Theorem 7,

$$x+y : x :: 5 : 2;$$

Theorem 4, $y : x :: 3 : 2$;

$$\therefore 3x=2y, \text{ or } y=\frac{3x}{2};$$

substituting this value of y in the second equation,

$$x^2+\frac{3x^2}{2}=40, \text{ or } 2x^2+3x^2=80, \text{ or } x^2=16;$$

$$\therefore x=\pm 4, \text{ and } y=\frac{3x}{2}=\pm 6.$$

8. Given $\overline{x+y}^2 : \overline{x-y}^2 :: 36 : 1$, and $x^2-y^2=24$, } to find the values of x and y.

From the first equation, by Theorem 12,

$$x+y : x-y :: 6 : 1;$$

Theorem 5, $2x : 2y :: 7 : 5$;

Theorem 9, $x : y :: 7 : 5$;

$\therefore y=\frac{5x}{7}$, or $y^2=\frac{25x^2}{49}$;

substituting this value of y^2 in the second equation,

$$x^2-\frac{25x^2}{49}=24\,;\ \therefore\ 49x^2-25x^2=1176,$$
$$\text{or } 24x^2=1176\,;\ \therefore\ x^2=49\,;$$

whence, $x=\pm 7$, and $y=\pm 5$.

9. Given $2x^2+2y^2=\frac{122}{x-y}$, and $\frac{xy}{2}=\frac{30}{2(x-y)}$, } to find the values of x and y.

Dividing the first equation by 2,

$$x^2+y^2=\frac{61}{x-y}\,;\ (\text{A})$$

multiplying the second equation by 4,

$$2xy=\frac{60}{x-y}\,;\ (\text{B})$$

subtracting this equation from equation (A),

$$x^2-2xy+y^2=\frac{1}{x-y},\ \text{or}\ (x-y)^2=\frac{1}{x-y}\,;$$
$$\therefore\ (x-y)^3=1,\ \text{or}\ x-y=1\,;$$

substituting this value of $x-y$, in the equations (A and B),

$$x^2+y^2=61,$$
$$\text{and}\quad 2xy=60\,;$$

by addition, $x^2+2xy+y^2=121$;

$$\therefore\ x+y=\pm 11\,;$$
$$\text{but}\quad x-y=\quad 1\,;$$

by addition, $2x\quad =12$, or -10;

$$\therefore\ x=\ 6,\ \text{or}\ -\ 5\,;$$

but by subtraction, $2y=10$, or -12;

$$\therefore\ y=\ 5,\ \text{or}\ -\ 6.$$

10. Given $\frac{5+\sqrt{25-x^2}}{5-\sqrt{25-x^2}}=9$, to find the values of x.

The value of a fraction not being altered by multiplying the numerator and denominator by the same number; therefore, multiply the numerator and denominator of the fractional part by the numerator, and the equation becomes $\frac{\overline{5+\sqrt{25-x^2}}^2}{x^2}=9$;

extracting the square root, $\frac{5+\sqrt{25-x^2}}{x}=\pm 3$;

$\therefore\ 5+\sqrt{25-x^2}=\pm 3x$;

by transposition, $\sqrt{25-x^2}=\pm 3x-5$;

squaring each side, $25-x^2=9x^2\pm 30x+25$;

$\therefore\ 10x^2=\pm 30x$; whence, $x=\pm 3$.

EXAMPLES FOR PRACTICE

In Pure Quadratics, &c. of one and two unknown Quantities.

1. Given $4x^2-35=3x^2-10$, to find the values of x.

Ans. $x=\pm 5$.

2. Given $2x^2+6=\frac{5x^2+27}{3}$, to find the values of x.

Ans. $x=\pm 3$.

3. Given $x : y :: 4 : 3$, and $xy=108$, } to find the values of x and y.

Ans. $\begin{cases} x=\pm 12. \\ y=\pm\ 9. \end{cases}$

4. Given $x+y : x-y :: 7 : 1$, and $x^2-y^2=63$, } to find the values of x and y.

Ans. $\begin{cases} x=\pm 12. \\ y=\pm\ 9. \end{cases}$

5. Given $x^2-y^2 : x^2+y^2 :: 5 : 13$, and $xy^{\frac{1}{2}}=12$, } to find the values of x and y.

Ans. $\begin{cases} x=6. \\ y=4. \end{cases}$

6. Given $x^2+2xy+y^2 : (x-y)^2 :: 16 : 1$, and $3x^2-2y^2=57$, } to find the values of x and y.

Ans. $\begin{cases} x=\pm 5. \\ y=\pm 3. \end{cases}$

7. Given $x^2-y^2 : x+y :: 2 : 1$, and $x^2+2xy+y^2=64$, } to find the values of x and y.

Ans. $\begin{cases} x=5 \text{ or } -3. \\ y=3 \text{ or } -5. \end{cases}$

8. Given $x^2-xy : xy-y^2 :: 3 : 2$, and $\frac{x-y}{2}=\frac{1}{2}$, } to find the values of x and y.

Ans. $\begin{cases} x=3. \\ y=2. \end{cases}$

9. Given $x^3-y^3 : x^2y-xy^2 :: 7 : 2$, and $x+y=6$, } to find the values of x & y.

Ans. $\begin{cases} x=4 \text{ or } 2. \\ y=2 \text{ or } 4. \end{cases}$

10. Given $x^3-y^3=27\frac{1}{2}$, and $x^2y-xy^2=\frac{35}{4}$, } to find the values of x and y.

Ans. $\begin{cases} x=3\frac{1}{2} \text{ or } -2\frac{1}{2}. \\ y=2\frac{1}{2} \text{ or } -3\frac{1}{2}. \end{cases}$

11. Given $x^2+y=\frac{657}{y}$, and $x+y^{\frac{1}{2}}=\frac{33}{y^{\frac{1}{2}}}$, } to find the values of x and y.

Ans. $\begin{cases} x=8 \text{ or } \frac{9}{\sqrt{24}}. \\ y=9 \text{ or } 24. \end{cases}$

12. Given $\frac{1}{x}+\frac{1}{y}=\frac{3}{8}$, and $\frac{1}{xy}=\frac{1}{32}$, } to find the values of x and y.

Ans. $\begin{cases} x=4 \text{ or } -8. \\ y=8 \text{ or } -4. \end{cases}$

13. Given $x^{\frac{2}{3}}+2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}=25$, and $x+y=35$, } to find the values of x and y.

Ans. $\begin{cases} x=27, \text{ or } -8. \\ y=8, \text{ or } -27. \end{cases}$

14. Given $x^6+6xy.(x^4+y^4)+4x^3y^3+y^6=46656-12x^2y^2.(x^2+xy+y^2)$, and $x-y=2$, to find the values of x and y.

Ans. $\begin{cases} x=4, \text{ or } -2. \\ y=2, \text{ or } -4. \end{cases}$

15. Given $x+\sqrt{27+x^2}=\frac{54}{\sqrt{27+x^2}}$, to find the values of x.

Ans. $x=\pm 3$.

16. Given $\frac{10-\sqrt{100-x^2}}{10+\sqrt{100-x^2}}=\frac{1}{9}$, to find the values of x.

Ans. $x=\pm 6$.

17. Given $x+y+\sqrt{x^2+y^2}=\frac{xy}{2}$, and $x^2+y^2=\overline{x+y}|^2-\frac{1}{4}xy^2$, } to find the values of x and y.

Ans. $\begin{cases} x=6. \\ y=8. \end{cases}$

18. Given $x^2-y^2=133$, and $xy-y^2=42$, } to find the values of x and y.

Ans. $\begin{cases} x=\pm 13. \\ y=\pm 6. \end{cases}$

19. Given $x^3-y^3=152$, and $x-y=2$, } to find the values of x and y.

Ans. $\begin{cases} x=6, \text{ or } -4. \\ y=4, \text{ or } -6. \end{cases}$

20. Given $x^4+xy^3+x^3y+y^4=8301256384$, and $x^4-x^3y+x^2y^2-xy^3+y^4=2159536$, } to find the values of x and y.

Ans. $\begin{cases} x=20. \\ y=42. \end{cases}$

EXAMPLES FOR PRACTICE

In the application of Pure Quadratics, &c. of one and two unknown quantities.

NOTE.—The application of Pure Quadratics to the solution of problems, is precisely similar to the application of Simple Equations to the solution of problems already given.

1. Bought a piece of cloth for £26. 9s., at as many shillings per yard as the piece contained yards. Required its length.

Ans. 23 yards.

2. Find two numbers, the sum of whose squares is to their difference as 5 to 3, and whose product is 18.

Ans. 6 and 3.

3. A farmer has two square fields, the length of a side of the greater is 5 chains more than a side of the less, and their areas are in the proportion of 9 to 16. Required the area of each.

Ans. $\begin{cases} 22a.\ 2r. \\ 40a.\ 0r. \end{cases}$

4. Bought a piece of cloth and a piece of silk for £9, and paid as many shillings per yard for the cloth as there were yards of silk, and for the silk as many shillings per yard as there were yards of cloth, and the square of the number of yards of silk is to the square of the number of yards of cloth as 25 to 4. Required the number of yards of each.

Ans. 15 yards of silk, and 6 yards of cloth.

5. Upon a rectangular piece of building ground, whose length is to its breadth as 5 to 3, there is to be a house erected upon the end of it the whole width of the ground, to take up one fifth of the whole, and to leave 300 square yards for a garden. Required the dimensions of the house and garden.

Ans. The house 15 yards by 5, and the garden 20 yards by 15.

6. Divide the number 75 into two such parts, that the quotient of the less divided by the greater, may be to the quotient of the greater divided by the less, as $\frac{7}{8}$ to $\frac{8}{7}$.

Ans. 35 and 40.

7. A company of merchants built a ship, which cost £5000, and divided their interest therein into a certain number of shares, portioning to each as many shares as there were merchants in company; the square of the number of merchants, is to $\frac{3}{5}$ of the number of shares $+5$, as 5 to 4. Required the number of merchants, and the sum contributed by each.

Ans. 5 merchants, and £1000 the sum contributed by each.

8. Bought a number of yards of silk for £3. 16s., and a quantity of cloth, which was in proportion to the silk as 4 to 5: I found that 5 yards of cloth cost

me as much as 8 yards of silk, and the number of shillings that one yard of cloth cost me, was to five-sixths of the number of yards of cloth as 4 to 5. How many yards of each did I buy?

Ans. 12 yards of cloth, and 15 of silk.

9. Find two numbers whose sum is to the difference of their squares as 1 to 3, and four times their product is 72.

Ans. 6 and 3.

10. Find two numbers the sum of whose reciprocals is $\frac{1}{2}$, and the sum of the reciprocals of their squares is $\frac{5}{36}$.

Ans. 3 and 6.

11. The difference of the reciprocals of two numbers is $\frac{2}{15}$, and the difference of the numbers multiplied by twice their product is equal to 60. Required the numbers.

Ans. 5 and 3.

12. The difference of the cubes of two numbers is 98, and the difference of the numbers multiplied by their product is 30. Required the numbers.

Ans. 5 and 3.

13. There is a number consisting of two digits, which being multiplied by the right-hand digit, the product is 175, but if the number be divided by the right-hand digit, the quotient will be 7. Required the number.

Ans. 35.

15. Five boys, A, B, C, D, and E, played at marbles; A won a certain number, B twice as many, C a number equal to twice the product of A's and B's, D as many as C, twice as many as B and one over, and E won a number equal to the square root of D's: the whole number won was 401 + 5 times A's added to D's. Required the number won by each.

Ans. A won 10, B 20, C 400, D 441, and E 21.

16. Required four numbers in the proportion of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{6}$, the sum of whose squares is 585.

Ans. 18, 12, 9, and 6.

17. Two merchants, A and B, entered into partnership; A's stock was in proportion to B's as 4 to 5; at the expiration of one year, A found his share of the gain to be in proportion to one-eight of his stock as 16 to 25: they continued their trade during as many years as were equal to one-fifth of the number of pounds which B contributed more than A, and at the end of that time found their whole gain to be £1440. What did each contribute to the stock, and how many years did they continue in business?

Ans. A's stock £400, B's stock £500, and they continued their business 20 years.

18. The sum of two numbers is 20, and the sum of their squares 272. Required the numbers.

Ans. 4 and 16.

19. Required the side of a tetraedron, the solidity of which is equal to that of a cube whose side is 2.

Ans. 1.0793.

20. The difference of the sides of two cubical blocks of marble is 2 feet, and the difference of their contents 218 cubical feet. Required a side of each.

Ans. 7 and 5.

ADFECTED QUADRATIC EQUATIONS.

An Adfected Quadratic Equation is that which contains both the unknown quantity and its square. Thus, $x^2+6x+6=22$, and $4x^2+5x=26$ are Adfected Quadratic Equations. But the Author of the present Treatise has considered all equations as adfected quadratics, in which there are two terms involving the unknown quantity, or any function of it, where the index of one is double that of the other. Thus, $2x^4-4x^2=16$, $x^6-x^3=a$, $x-x^{\frac{1}{2}}=b$, $x^{\frac{4}{3}}+x^{\frac{2}{3}}=c$, $x+4+\sqrt{x+4}=12$, are considered Adfected Quadratic Equations.

Also, in equations where more than one unknown quantity is found forming two terms, one term being square of the other. Thus, $(x+y)^2+(x+y)=a$, or $x+y+\sqrt{x+y}=b$ is considered an Adfected Quadratic Equation. In the first of these equations, the first term, $(x+y)^2$, is square of the second term, $(x+y)$. Also, in the second equation, the first term, $x+y$ is square of the second term, $\sqrt{x+y}$.

RULE.

1. *Transpose the unknown quantities to one side of the equation, and the known quantities to the other, observing to begin with the highest power of the unknown quantity. Then, if the first term contain a coefficient, divide the whole equation by that coefficient.*

2. *Complete the square, by adding the square of half the coefficient of the second term to each side of the equation.*

3. *Extract the square root of each side of the equation, and the unknown quantity will be easily determined as follows:—*

1. In the first example, viz. $x^2+6x+6=22$.

2. Transpose the 6, and the equation becomes $x^2+6x=16$.

3. Here, the unknown quantity in the first term is square of the unknown quantity in the second term.

4. Then 6 being the coefficient of the second term, half of which is 3, this being squared and added to each side of the equation, it becomes $x^2+6x+9=16+9=25$.

5. Extracting the square root of each side of this equation, and it becomes $x+3=\pm 5$;

6. by transposition, $x=\pm 5-3=2$, or -8.

In the second example, $4x^2+5x=26$.

1. Here, the first term has a coefficient, 4.

2. Divide the whole equation by 4, and it becomes $x^2+\frac{5}{4}x=\frac{26}{4}$.

3. Then $\frac{5}{4}$ being the coefficient of the second term, half of which is $\frac{5}{8}$, this being squared and added to each side of the equation, it becomes $x^2+\frac{5}{4}x+\frac{25}{64}=\frac{26}{4}+\frac{25}{64}=\frac{441}{64}$.

4. Extracting the square root of each side of this equation, $x+\frac{5}{8}=\pm\frac{21}{8}$.

5. By transposition, $x=\pm\frac{21}{8}-\frac{5}{8}=2$, or $-\frac{26}{8}$.

In the third example, $2x^4-4x^2=16$.

1. Here, the index of the unknown quantity in the first term is double that of the second, and the first term has a coefficient.
2. Divide by 2, the coefficient of the first term, $x^4-2x^2=8$.
3. Complete the square, $x^4-2x^2+1=8+1=9$.
4. Extract the square root, $x^2-1=\pm 3$.
5. By transposition, $x^2=\pm 3+1=4$, or -2.
6. Extracting the square root, $x=\pm 2$, or $\pm\sqrt{-2}$.

In the example, $x+4+\sqrt{x+4}=12$.

1. Here, $x+4$ is considered the first term, and $\sqrt{x+4}$ the second term, $\therefore$ the first term is square of the second.
2. One being the coefficient of the second term, half of which is $\frac{1}{2}$.
3. Completing the square, $x+4+\sqrt{x+4}+\frac{1}{4}=12+\frac{1}{4}=\frac{49}{4}$.
4. Extracting the square root, $\sqrt{x+4}+\frac{1}{2}=\frac{7}{2}$.
5. By transposition, $\sqrt{x+4}=\frac{7}{2}-\frac{1}{2}=3$.
6. Squaring each side, $x+4=9$.
7. By transposition, $x=5$.

Adfected Quadratics of one unknown quantity with their solutions.

1. Given $3x^2+6x+6=78$, to find the values of x.

By transposition, $3x^2+6x=78-6=72$;
dividing by 3, $x^2+2x=24$;
completing the square, $x^2+2x+1=25$;
extracting the root, $x+1=\pm 5$;
$\therefore x=\pm 5-1=4$, or -6.

2. Given $x^2-\frac{4x+6}{7}=8x+3$, to find the values of x.

Multiplying the equation by 7,
$7x^2-4x-6=56x+21$;
by transposition, $7x^2-60x=27$;
dividing by 7, $x^2-\frac{60}{7}x=\frac{27}{7}$;
completing the square, $x^2-\frac{60}{7}x+\left|\frac{30}{7}\right|^2=\frac{27}{7}+\frac{900}{49}=\frac{1089}{49}$;
extracting the root, $x-\frac{30}{7}=\pm\frac{33}{7}$;
$\therefore x=\pm\frac{33}{7}+\frac{30}{7}=9$, or $-\frac{3}{7}$.

3. Given $y-y^{\frac{1}{2}}+5=8+y^{\frac{1}{2}}$, to find the value of y.

By transposition, $y-2y^{\frac{1}{2}}=3$;
completing the square, $y-2y^{\frac{1}{2}}+1=4$;
extracting the root, $y^{\frac{1}{2}}-1=2$;
by transposition, $y^{\frac{1}{2}}=3$; $\therefore y=9$.

4. Given $x^3-20x^{\frac{3}{2}}=189$, to find the values of x.

completing the square, $x^3-20x^{\frac{3}{2}}+100=289$;

extracting the root, $x^{\frac{3}{2}}-10=\pm 17$;

by transposition, $x^{\frac{3}{2}}=\pm 17+10=27$, or -7;

$\therefore\ x=\overline{27}|^{\frac{2}{3}}=9$, or $\overline{-7}|^{\frac{2}{3}}$.

5. Given $2x+4=\sqrt{2x+4}+12$, to find the values of x.

By transposition, $2x+4-\sqrt{2x+4}=12$;

completing the square, $2x+4-\sqrt{2x+4}+\frac{1}{4}$

$=12+\frac{1}{4}=\frac{49}{4}$;

extracting the root, $\sqrt{2x+4}-\frac{1}{2}=\frac{7}{2}$;

$\therefore\ \sqrt{2x+4}=4$;

squaring each side, $2x+4=16$;

by transposition, $2x=12$; $\therefore\ x=6$.

6. Given $\frac{2x}{4+x}+\frac{3}{6-x}=\frac{8x+1}{4x+4}$, to find the values of x.

Multiplying the equation by $4+x$,

$$2x+\frac{12+3x}{6-x}=\frac{8x^2+33x+4}{4x+4};$$

multiplying this equation by $4x+4$,

$$8x^2+8x+\frac{12x^2+60x+48}{6-x}=8x^2+33x+4;$$

(Def. 7.) by transposition, $\frac{12x^2+60x+48}{6-x}=25x+4$;

multiplying this equation by $6-x$,

$$12x^2+60x+48=146x-25x^2+24;$$

by transposition, $37x^2-86x=-24$;

completing the square, $x^2-\frac{86}{37}x+\left|\frac{43}{37}\right|^2=\frac{1849}{1369}-\frac{24}{37}=\frac{961}{1369}$;

extracting the root, $x-\frac{43}{37}=\pm\frac{31}{37}$;

$\therefore\ x=\pm\frac{31}{37}+\frac{43}{37}=2$, or $\frac{12}{37}$.

7. Given $\frac{x+\sqrt{x^2-4}}{x-\sqrt{x^2-4}}=\overline{2x-3}|^2$, to find the values of x.

Multiplying the numerator and denominator of the fractional part by $x+\sqrt{x^2-4}$,

$$\frac{(x+\sqrt{x^2-4})^2}{4}=\overline{2x-3}|^2;$$

extracting the square root of each side,

$$\frac{x+\sqrt{x^2-4}}{2}=\pm(2x-3).$$

Taking the positive quantity, and multiplying by 2,

$$x+\sqrt{x^2-4}=4x-6;$$

by transposition, and squaring each side,

$$x^2-4=9x^2-36x+36;$$

by transposition, $8x^2-36x=-40$;

dividing by 8, and completing the square,

$$x^2-\frac{9}{2}x+\frac{81}{16}=\frac{81}{16}-5=\frac{1}{16};$$

extracting the root, $x-\frac{9}{4}=\pm\frac{1}{4}$;

$$\therefore\ x=\pm\frac{1}{4}+\frac{9}{4}=\frac{5}{2},\text{ or }2.$$

But if the negative value be taken,

$$\frac{x+\sqrt{x^2-4}}{2}=-2x+3;$$

$$\therefore\ x+\sqrt{x^2-4}=-4x+6;$$

by transposition, $\sqrt{x^2-4}=-5x+6$;

squaring each side, $x^2-4=25x^2-60x+36$;

by transposition, $24x^2-60x=-40$;

dividing by 24, $x^2-\frac{60}{24}x=-\frac{5}{3}$;

completing the square, $x^2-\frac{60}{24}x+\frac{900}{576}=\frac{900}{576}-\frac{5}{3}=-\frac{60}{576}$;

extracting the square root, $x-\frac{30}{24}=\pm\frac{\sqrt{-60}}{24}$;

$$\therefore\ x=\frac{30\pm\sqrt{-60}}{24}.$$

8. Given $8x-2x^2+4\sqrt{2x^2-8x+15}=18$, to find the values of x.

Subtracting 15 from each side of the equation,

$$8x-2x^2-15+4\sqrt{2x^2-8x+15}=3;$$

changing the signs, $2x^2-8x+15-4\sqrt{2x^2-8x+15}=-3$;

completing the square, $2x^2-8x+15-4\sqrt{2x^2-8x+15}+4=4-3=1$;

extracting the root, $\sqrt{2x^2-8x+15}-2=\pm1$.

Let the positive value be taken, then by transposition, $\sqrt{2x^2-8x+15}=1+2=3$;
squaring each side, $2x^2-8x+15=9$;
by transposition, $2x^2-8x=-6$.
dividing by 2, $x^2-4x=-3$;
completing the square, $x^2-4x+4=1$;
extracting the root, $x-2=\pm 1$;
$\therefore$ by transposition, $x=\pm 1+2=3$, or 1.

But if the negative value be taken, viz.

$$\sqrt{2x^2-8x+15}-2=-1;$$

by transposition, $\sqrt{2x^2-8x+15}=1$;
squaring both sides, $2x^2-8x+15=1$;
by transposition, and dividing by 2,
$x^2-4x=-7$;
completing the square, $x^2-4x+4=-3$;
extracting the root, $x-2=\pm\sqrt{-3}$;
by transposition, $x=2\pm\sqrt{-3}$.

9. Given $x^4-2x^3+x=870$, to find the values of x.

Here, adding and subtracting x^2, and the equation becomes $x^4-2x^3+x^2-x^2+x=870$;

$$\therefore (x^2-x)^2-(x^2-x)=870;$$

completing the square, $(x^2-x)^2-(x^2-x)+\frac{1}{4}$
$=870+\frac{1}{4}=\frac{3481}{4}$;

extracting the root, $x^2-x-\frac{1}{2}=\pm\frac{59}{2}$;

$$\therefore x^2-x=\pm\frac{59}{2}+\frac{1}{2}=30, \text{ or } -29.$$

Let the affirmative value be taken, and complete the square, $x^2-x+\frac{1}{4}=30+\frac{1}{4}=\frac{121}{4}$;

extracting the root, $x-\frac{1}{2}=\pm\frac{11}{2}$;

$$\therefore\ x=\pm\frac{11}{2}+\frac{1}{2}=6,\text{ or }-5.$$

But if $x^2-x=-29$;

comp. the sq. $x^2-x+\frac{1}{4}=\frac{1}{4}-29=-\frac{115}{4}$;

extracting the root, $x-\frac{1}{2}=\pm\frac{\sqrt{-115}}{2}$;

$$\therefore\ x=\frac{1\pm\sqrt{-115}}{2}.$$

10. Given $2x^2+\sqrt{18x^2+9x}=54-x$, to find the value of x.

By transposition, $2x^2+x+\sqrt{18x^2+9x}=54$;

$\therefore$ (Def. 14.) $2x^2+x+3\sqrt{2x^2+x}=54$;

completing the square, $2x^2+x+3\sqrt{2x^2+x}+\frac{9}{4}=54+\frac{9}{4}=\frac{225}{4}$;

extracting the root, $\sqrt{2x^2+x}+\frac{3}{2}=\pm\frac{15}{2}$;

$\therefore\ \sqrt{2x^2+x}=6$, or -9;

squaring each side, $2x^2+x=36$, or 81;

$$\therefore\ x^2+\frac{1}{2}x=18,\text{ or }\frac{81}{2}.$$

Let the first value be taken, and complete the square, $x^2+\frac{1}{2}x+\frac{1}{16}=18+\frac{1}{16}=\frac{289}{16}$;

extracting the root, $x+\frac{1}{4}=\pm\frac{17}{4}$;

$\therefore\ x=4$, or $-\frac{9}{2}$.

But if $x^2+\frac{1}{2}x=\frac{81}{2}$;

complete the square, $x^2+\frac{1}{2}x+\frac{1}{16}=\frac{81}{2}+\frac{1}{16}$

$=\frac{649}{16}$;

extracting the root, $x+\frac{1}{4}=\pm\frac{\sqrt{649}}{4}$;

$$\therefore x=\frac{-1\pm\sqrt{649}}{4}.$$

EXAMPLES FOR PRACTICE

In Adfected Quadratics of one unknown quantity.

1. Given $x^2+2x=15$, to find the values of x.

Ans. $x=3$, or -5.

2. Given $x^2+6x=40$, to find the values of x.

Ans. $x=4$, or -10.

3. Given $4x^2-8x=60$, to find the values of x.

Ans. $x=5$, or -3.

4. Given $y^2-y=30$, to find the values of y.

Ans. $y=6$, or -5.

5. Given $y-y^{\frac{1}{2}}=6$, to find the value of y.

Ans. $y=9$.

6. Given $y-\frac{1}{2}\sqrt{y}=14$, to find the value of y.

Ans. $y=16$.

7. Given $y+2y^{\frac{1}{2}}=7+\frac{1}{2}\sqrt{y}$, to find the value of y.

Ans. $y=4$.

8. Given $3x^2+9x+6=36$, to find the values of x.

Ans. $x=2$, or -5.

9. Given $4x^2-x=60$, to find the values of x.

Ans. $x=4$, or $-\frac{15}{4}$.

10. Given $x+4+\frac{3x+6}{x}=12$, to find the values of x.

Ans. $x=3$, or 2.

11. Given $\frac{x+6}{4}+\frac{3x-4}{x+1}=5$, to find the values of x.

Ans. $x=6$, or -5.

12. Given $x^4+6x^2=40$, to find the values of x.

Ans. $x=\pm 2$, or $\pm\sqrt{-10}$.

13. Given $x^3+\frac{x^3+128}{x^3}=67$, to find the values of x.

Ans. $x=4$, or $\sqrt[3]{2}$.

14. Given $\frac{x-4}{x}+\frac{4x-5}{x^2}=\frac{20}{5x}$, to find the values of x.

Ans. $x=5$, or -1.

15. Given $\frac{2x+10}{9}+\frac{3x-4}{x+4}=6-\frac{12x+6}{18}$, to find the values of x.

Ans. 4, or $-\frac{25}{4}$.

16. Given $\frac{25}{x-1}+\frac{12}{x}=7$, to find the values of x.

Ans. $x=6$, or $\frac{2}{7}$.

17. Given $\frac{4x+7}{2x}+\frac{6x-9}{2x-4}=\frac{12x+56}{4x}$, to find the values of x.

Ans. $x=3\frac{1}{2}$, or 3.

18. Given $\frac{3x+13}{x+1}-\frac{6x-\frac{x^2-1}{x+1}}{2x+5}=2$, to find the values of x.

Ans. $x=9$, or -2.

19. Given $\sqrt{x^2-6x+2}=3$, to find the values of x.

Ans. $x=7$, or -1.

20. Given $\sqrt{3x+1}\times\sqrt{2x-1}=12$, to find the values of x.

Ans. $x=5$, or $-\frac{58}{12}$.

21. Given $\frac{5x^2-12}{x+6}-\frac{3x^2}{9}=\frac{14x-2x^2}{6}$, to find the values of x.

Ans. $x=6$, or $-\frac{3}{4}$.

22. Given $\frac{12x}{\sqrt{x^3}}+\frac{24+7x}{\sqrt{x}}=\frac{5x+846}{x^{\frac{1}{2}}}$, to find the values of x.

Ans. $x=9$, or $-\frac{94}{7}$.

23. Given $x^{\frac{4}{3}}-6x^{\frac{2}{3}}=27$, to find the values of x.

Ans. $x=27$, or $-\overline{3|}^{\frac{3}{2}}$.

24. Given $x^{\frac{1}{3}}+4x^{\frac{1}{6}}=16-\sqrt[3]{x}$, to find the values of x.

Ans. $x=64$, or $-\overline{4|}^{6}$.

25. Given $\sqrt{x^7}-\frac{36}{\sqrt{x}}=5\sqrt{x^3}$, to find the values of x.

Ans. $x=\pm 3$, or $\pm\sqrt{-4}$.

26. Given $x^2+4x+4\sqrt{x^2+4x+15}=45$, to find the values of x.

Ans. $x=3$, or -7, or $\pm\sqrt{89}-2$.

27. Given $x^4-4x^3+8x=165$, to find the values of x.

Ans. $x=5$, or -3, or $1\pm\sqrt{-10}$.

28. Given $6x-3x^2+6\sqrt{3x^2-6x-8}=-24$, to find the values of x.

Ans. $x=6$, or -4, or $1\pm\sqrt{5}$.

29. Given $\overline{2x-\frac{12}{x}|}^{2}-30+2x=\frac{12}{x}$, to find the values of x.

Ans. $x=4$, or $-\frac{3}{2}$, or $\frac{-3\pm\sqrt{33}}{2}$.

30. Given $x^4+2x^3-x=3080$, to find the values of x.

Ans. $x=7$, or -8, or $\frac{1\pm\sqrt{-219}}{2}$.

31. Given $3x^n.\sqrt[3]{x^n}+\frac{4x^n}{\sqrt[3]{x^n}}=4$, to find the value of x.

Ans. $\left(\frac{2}{3}\right)^{\frac{3}{2n}}$.

32. Given $x^4+10x^3+54x^2+145x=602$, to find the values of x.

Ans. $x=2$, or -7, or $\frac{-5\pm\sqrt{-147}}{2}$.

ADFECTED QUADRATIC EQUATIONS

Of two unknown quantities with their solutions.

1. Given $\left.\begin{matrix}2x-y=15,\\ \text{and } 2x-y^2=9,\end{matrix}\right\}$ to find the values of x and y.

From the second equation, $x=\frac{9+y^2}{2}$;

substituting this value of x in the first equation,

$9+y^2-y=15$;

by transposition, $y^2-y=6$;

completing the square, $y^2-y+\frac{1}{4}=6+\frac{1}{4}=\frac{25}{4}$;

extracting the root, $y-\frac{1}{2}=\pm\frac{5}{2}$;

by transposition, $y=3$, or -2;

$\therefore\ x=9$, or $\frac{13}{2}$.

Or thus:

From the first equation, $y=2x-15$;

substituting this value of y in the second equation, $4x^2-62x=-234$;

completing the square, $x^2-\frac{62}{4}x+\left|\frac{\overline{31}}{4}\right|^2=\frac{961}{16}$

$$-\frac{234}{4}=\frac{25}{16};$$

extracting the root, $x-\frac{31}{4}=\pm\frac{5}{4}$;

by transposition, $x=9$, or $\frac{13}{2}$;

$$\therefore\ y=3,\ \text{or}\ -2.$$

2. Given $4x+4y=56$, and $6x+3y=xy+12$, } to find the values of x and y.

Dividing the first equation by 4,

$$x+y=14;\ \therefore\ y=14-x;$$

substituting this value of y in the second equation, $6x+42-3x=14x-x^2+12$;

by transposition, $x^2-11x=-30$;

completing the square, $x^2-11x+\frac{121}{4}=\frac{121}{4}-30=\frac{1}{4}$;

extracting the root, $x-\frac{11}{2}=\pm\frac{1}{2}$;

by transposition, $x=6$, or 5;

$$\therefore\ y=8,\ \text{or}\ 9.$$

3. Given $x^2+y^2=50-x-y$, and $\frac{xy}{2}=10$, } to find the values of x and y.

Transposing the first equation,

$$x^2+y^2+x+y=50;$$

but, $2xy\quad\ =40$; (4 times 2nd. eq.)

by addition, $x^2+2xy+y^2+x+y=90$;

or $(x+y)^2+(x+y)=90$;

completing the square, $(x+y)^2+(x+y)+\frac{1}{4}=90+\frac{1}{4}=\frac{361}{4}$;

extracting the root, $x+y+\frac{1}{2}=\pm\frac{19}{2}$;

by transposition, $x+y=9$, or -10;

squaring both sides of this equation,

$x^2+2xy+y^2=81$, or 100;

but $4xy=80$,

by subtraction, $x^2-2xy+y^2=1$, or 20;

extracting the square root,

$x-y=\pm 1$, or $\pm\sqrt{20}$;

but $x+y=9$, or -10;

by addition, $2x=10$, or 8, or $-10\pm 2\sqrt{5}$;

$\therefore x=5$, or 4, or $-5\pm\sqrt{5}$;

but by subtraction, $2y=8$, or 10, or $-10\mp 2\sqrt{5}$;

$\therefore y=4$, or 5, or $-5\mp\sqrt{5}$.

4. Given $(x+y)^2+2x=195-2y$, and $x^2+y^2=97$, } to find the values of x and y.

Transposing the first equation, $(x+y)^2+2.(x+y)=195$;

completing the square, $(x+y)^2+2.(x+y)+1=196$;

extracting the root, $x+y+1=\pm 14$;

by transposition, $x+y=\pm 14-1=13$, or -15;

squaring each side, $x^2+2xy+y^2=169$, or 225;

but $x^2+y^2=97$;

by subtraction, $2xy=72$, or 128;

$\therefore\ 4xy\ \ =144,$ or 256;

but $x^2+2xy+y^2=169,$ or 225;

by subtraction, $x^2-2xy+y^2=25,$ or -31;

$\therefore\ x-y=\pm 5,$ or $\pm\sqrt{-31}$;

but $x+y=13,$ or -15;

by addition, $2x=18,$ or $8,$ or $-15\pm\sqrt{-31}$;

$\therefore\ x=9,$ or $4,$ or $\dfrac{-15\pm\sqrt{-31}}{2}$;

but by subtraction, $2y=8,$ or $18,$ or $\mp\sqrt{-31}-15$;

$\therefore\ y=4,$ or $9,$ or $\dfrac{\mp\sqrt{-31}-15}{2}$.

5. Given $2x^4y^2-4x^2y=4416,$ and $x^2+y=\dfrac{60}{y}-1,$ } to find the values of x and y.

Dividing the first equation by 2, and completing the square, $x^4y^2-2x^2y+1=2208+1=2209$;

extracting the root, $x^2y-1=\pm 47$;

by transposition, $x^2y=48,$ or -46;

multiplying the second equation by y,

$x^2y+y^2=60-y$;

substituting the affirmative value of x^2y, into this equation, $48+y^2=60-y$;

by transposition, $y^2+y=12$;

completing the square, $y^2+y+\dfrac{1}{4}=12+\dfrac{1}{4}=\dfrac{49}{4}$;

extracting the root, $y+\dfrac{1}{2}=\pm\dfrac{7}{2}$;

$\therefore\ y=3,$ or $-4.$

But if the negative value of x^2y be substituted,

$-46+y^2=60-y$;

by transposition, $y^2+y=106$;

completing the square, $y^2+y+\frac{1}{4}=106+\frac{1}{4}=\frac{425}{4}$;

extracting the root, $y+\frac{1}{2}=\pm\frac{\sqrt{425}}{2}$;

$$\therefore\ y=\frac{-1\pm5\sqrt{17}}{2}.$$

EXAMPLES FOR PRACTICE

In Adfected Quadratics of two unknown quantities.

1. Given $x^2-xy=60$, and $x+y=19$, } to find the values of x and y.

   Ans. $\begin{cases} x=12, \text{ or } -\frac{5}{2}. \\ y=7, \text{ or } 21\frac{1}{2}. \end{cases}$

2. Given $xy+18x=1728$, and $8y=9x$, } to find the values of x and y.

   Ans. $\begin{cases} x=32, \text{ or } -48. \\ y=36, \text{ or } -54. \end{cases}$

3. Given $5x+4y=35$, and $9y-xy=10x$, } to find the values of x and y.

   Ans. $\begin{cases} x=3, \text{ or } 21. \\ y=5, \text{ or } -17\frac{1}{2}. \end{cases}$

4. Given $2x+y=7$, and $x^2+3y=13$, } to find the values of x and y.

   Ans. $\begin{cases} x=2, \text{ or } 4. \\ y=3, \text{ or } -1. \end{cases}$

5. Given $xy+2y=28$, and $4xy+7y^2=192$, } to find the values of x and y.

Ans. $\begin{cases} x=5, \text{ or } -11\frac{1}{2}. \\ y=4, \text{ or } -2\frac{6}{7}. \end{cases}$

6. Given $6y-4x=x^2+xy+4$, and $8x-y=x(x+y)$, } to find the values of x and y.

Ans. $\begin{cases} x=2, \text{ or } \frac{2}{19}. \\ y=4, \text{ or } \frac{100}{133}. \end{cases}$

7. Given $40x-3xy=-4y$, and $3x+y+36=3y+2x$, } to find the values of x and y.

Ans. $\begin{cases} x=4, \text{ or } -12. \\ y=20, \text{ or } 12. \end{cases}$

8. Given $\frac{4x+3y}{y+2}-y=8-2y$, and $\frac{3x+4y}{8}=y$, } to find the values of x and y.

Ans. $\begin{cases} x=4, \text{ or } -\frac{128}{18}. \\ y=3, \text{ or } -\frac{32}{6}. \end{cases}$

9. Given $\frac{2x^3+x^2+1}{7}+4y=\frac{4x^3-y-1}{14}+13$, and $2x+4y=16$, } to find the values of x and y.

Ans. $\begin{cases} x=2, \text{ or } \frac{49}{4}. \\ y=3, \text{ or } -\frac{17}{8}. \end{cases}$

10. Given $\overline{x+y}|^2+2y=99-2x$, and $x^2+y^2=41$, } to find the values of x and y.

Ans. $\begin{cases} x=5, \text{ or } 4, \text{ or } \frac{-11\pm\sqrt{-39}}{2}. \\ y=4, \text{ or } 5, \text{ or } \frac{-11\mp\sqrt{-39}}{2}. \end{cases}$

11. Given $\frac{x^4}{y^2}+\frac{4x^2}{y}=5$, and $x^2+y^2=20$, } to find the values of x and y.

Ans. $\begin{cases} x=\pm 2, \text{ or } \pm\sqrt{-5}. \\ y=4, \text{ or } -5. \end{cases}$

12. Given $x+y+4\sqrt{x+y}=21$, and $xy=18$, } to find the values of x and y.

Ans. $\begin{cases} x=6, \text{ or } 3. \\ y=3, \text{ or } 6. \end{cases}$

13. Given $x+y+\sqrt{x+y}=12$, and $x^2+y^2=41$, } to find the values of x and y.

Ans. $\begin{cases} x=5, \text{ or } 4. \\ y=4, \text{ or } 5. \end{cases}$

14. Given $5x+3-\sqrt{5x-y+9}=24+y$, and $x+y=9$, } to find the values of x and y.

Ans. $\begin{cases} x=6, \text{ or } \frac{25}{6}. \\ y=3, \text{ or } \frac{29}{6}. \end{cases}$

15. Given $\frac{x}{2\sqrt{(x+y)^3}}+\frac{5\sqrt{x+y}}{x}=\frac{167\div 9}{2\sqrt{x+y}}$, and $x^2+y^2+x=46$, } to find the values of x and y.

Ans. $\begin{cases} x=5, \text{ or } -\frac{230}{41}. \\ y=4, \text{ or } -\frac{184}{41}. \end{cases}$

16. Given $x+y+\sqrt{\frac{x+y}{x-y}}=\frac{6}{x-y}$, and $x^2+y^2=41$, } to find the values of x and y.

Ans. $\begin{cases} x=\pm 5, \text{ or } \pm 3\sqrt{\frac{5}{2}}. \\ y=\pm 4, \text{ or } \pm \sqrt{\frac{37}{2}}. \end{cases}$

17. Given $x-\sqrt{x}=11-y$, and $36-x=y+8\sqrt{y-1}$, } to find the values of x and y.

Ans. $\begin{cases} x=4, \text{ or } \frac{196}{4225}. \\ y=9, \text{ or } \frac{45369}{4225}. \end{cases}$

18. Given $x^3-x^2+3x^2y-306x-2xy+3xy^2=y^2+306y-y^3$,

and $x-y+\sqrt{x-y}=6$, to find the values of x and y.

Ans. $\begin{cases} x=11, \text{ or } -6\frac{1}{2}. \\ y=7, \text{ or } -10\frac{1}{2}. \end{cases}$

19. Given $xy+\sqrt{x^2+y^2}=17$, and $x-y=1$, } to find the values of x and y.

Ans. $\begin{cases} x=4, \text{ or } -3. \\ y=3, \text{ or } -4. \end{cases}$

20. It is required to find four different values of x and y from the following equations:

$$5y+\frac{1}{5}.(x^2-15y-14)^{\frac{1}{2}}=\frac{1}{3}(x^2-108),$$

and $$\frac{x^3}{8y}+\frac{2}{3}x=x.\left(\frac{x}{3y}+\frac{1}{4}\right)^{\frac{1}{2}}.\ \frac{1}{2}y.$$

Answers.

$$x=12, \text{ or } -9\tfrac{1}{2}, \text{ or } \frac{25\pm\sqrt{41569}}{20}, \text{ or } \frac{45\pm\sqrt{3849}}{4}.$$

$$y=2, \text{or } -1\tfrac{7}{17}, \text{ or } \frac{25\pm\sqrt{41569}}{120}, \text{ or } \frac{135\pm3\sqrt{3849}}{8}.$$

Examples for Practice in Adfected Quadratics.

1. Find two numbers such, that their difference shall be 12, and their product 540.

Ans. 30 and 18.

2. Divide the number 80 into two such parts, that their product shall be 1375.

Ans. 25 and 55.

3. Bought two pieces of silk, one of which contained four yards less than the other, for £32. 16s., each piece cost as many shillings per yard as there were yards in the piece. Required the length of each.

Ans. 20 and 16 yards.

4. Bought a quantity of cloth for £30, and observe, if there had been 10 yards more, it would have cost me 3s. a yard less. Required the number of yards.

Ans. 40.

5. Required the length and breadth of that rectangular field, whose area is 1200 square yards, and if 4 yards be added to each of its dimensions, its area will be increased by 296 square yards.

Ans. 40 and 30 yards.

6. Bought two pieces of cloth for £44. 8s., the one contained eight yards more than the other, and the price of each piece, per yard, was three-fourths as many

shillings as the piece contained yards. Required the length of each piece and its price.

Ans. 20 yards at 15s., and 28 yards at 21s. per yard.

7. In taking the dimensions of a rectangular field, I found its diagonal to measure six chains more than its width, and three chains more than its length. Required its area.

Ans. 10*a*. 3*r*. 8*p*.

8. A Lady being asked her age answered, if 12 years be added to my age, and the square root of this sum extracted, it will equal one fourth of my age. Required her age.

Ans. 24.

9. A draper bought a quantity of cloth for £62. 10s. after parting with 10 yards, he sold the remainder for £55, and gained 2s. 6d. per yard. Required the number of yards bought, and the price per yard.

Ans. 50 yards at 25s. per yard.

10. A grocer bought two casks of sugar, one of which cost five shillings per cwt. more than the other; for the best quality he paid £55, and for the other, which contained 8 cwt. more, he paid £67. 10s. Required the quantity of each, and its price per cwt.

Ans. { 30 cwt. at 45s.
       { 22 cwt. at 50s.

11. A person bought 25 yards of silk, and 15 yards of Irish cloth, for £9: he had three yards more of Irish cloth for 12s. than he had of silk for 18s. Required the price of a yard of each.

Ans. Silk 6s., and cloth 2s. per yard.

12. A gravel walk is to be made of equal width round a rectangular garden, whose length is 57 and breadth

29 yards, which shall take up one fourth of the garden. What must be its width?

Ans. 2.554354 yards.

13. Divide the number 56 into two such parts, so as their product will be 559.

Ans. 13 and 43.

14. The temple of the three graces at Athens stood upon a rectangular area, the length of which exceeded its breadth by three paces, and the distance from one angle to that which was diametrically opposite to it, exceeded the length also by so many paces. Required its length, breadth, and diametrical distance.

Ans. Breadth 9, length 12, and the diagonal 15.

15. A mercer sold a certain quantity of lace for £4. 19s., and found if he had sold 4 yards less for the same money, he would have received a shilling per yard more than he did. Required the number of yards sold.

Ans. 22.

16. A farmer bought a flock of sheep, consisting of ewes and lambs; the number of lambs exceeded the number of ewes by 10: for each lamb he gave as many shillings as there were ewes, and for each ewe, as many shillings as there were lambs. He afterwards bought a number of ewes at the same rate, which number is to the number of ewes in the first flock as 7 to 4; he paid £7. 10s. less for the second flock than for the first. Required the number of ewes and lambs in the first flock, also the number of ewes in the second flock.

Ans. 20 ewes and 30 lambs in the first flock.
and 35 ewes in the second flock.

17. Required that number, which being divided by two thirds of the sum of the digits, the quotient will

be 9 ; but if the square of the right-hand digit be added to the number, the sum will be equal to the digits inverted plus 25.

Ans. 54.

18. Required two numbers whose product is 300, and if 10 be added to the less number, and 8 subtracted from the greater, the product of the sum and remainder shall also be equal to 300.

Ans. 20 and 15.

19. A grocer sold a quantity of sugar for £30, and by so doing gained four-fifths of the prime cost per cent. profit. Required the prime cost.

Ans. £25.

20. A began trade on the first of January, and on the first of May following he took in B as a partner, who brought into stock £200 ; at the expiration of 12 months from this transaction they dissolved partnership, when it was found that there had been gained, since A commenced, £300. A received for his stock and gain £500. Required the sum with which A commenced, and each person's share of the gain.

Ans. A began with £300, and his gain was £200 ; and B's gain was £100.

21. Two boys, A and B, comparing their marbles, found that the number which they both had, added to its square root, the sum would be 12, and the sum of their squares was 41. How many had each ?

Ans. A had 4 and B 5.

22. The plate of a looking-glass is 18 inches by 12, is to be framed with a frame of equal width, whose area is to be equal to half that of the glass. Required the width of the frame.

Ans. 1.6241438.

23. Find two numbers, the square of whose sum shall exceed twelve times their sum by 325, and that their product may exceed five times their sum by 31.

Ans. 12 and 13.

24. A and B set out from two towns, which were 120 miles distant, to meet each other; A travelled five miles a day, and the number of days, at the end of which they met, was greater by three than the number of miles which B went per day. How many miles did each go?

Ans. A 50 and B 70.

25. A mercer sold 50 yards of lace and 80 yards of silk for £40, and observed that his customer received 5 yards more of lace for £8, than he did of silk for £3. 15s. Required the price per yard of each.

Ans. Lace 8s. per yard, and silk 5s.

26. Two merchants, A and B, entered into partnership, by which they gained £120; A's money was three months in trade, and his share of the gain was £60 less than his stock: B's money, which was £50 more than A's, was in the trade five months. Required each person's stock.

Ans. A's £100, and B's £150.

27. It is required to find two numbers such, that if their sum be multiplied by the greater, the product will be 126, and whose difference, multiplied by the less, the product will be 20.

Ans. 9 and 5.

28. Bought a number of yards of silk for £7. 4s., and a quantity of Irish cloth, exceeding the silk by 16 yards, for an equal sum, but paid 1s. 6d. per yard less for the cloth th an for the silk. Required the number of yards of each.

Ans. 32 yards of silk, and 48 of cloth.

29. A person bought four remnants of silk, the number of yards in the first was in the same proportion to the yards in the second, that the third was to the fourth, and the number of yards in the first was to the number of yards in the fourth as 1 to 5, the number of yards in the second was to the number of yards in the third as $2\frac{1}{2}$ to $4\frac{1}{2}$, and the number of yards in the second and fourth was just 20. Required the length of each remnant.

Ans. 3, 5, 9, and 15 yards respectively.

30. Two merchants, A and B, received each £5940 per annum, in their respective trades; A by industry cleared 2 per cent. more than B, by which means he gained £100 per annum more than B did. Required each person's gain per cent. and the sum each realized in 10 years.

Ans. A gained 10 per cent. and realized £5400,
B gained 8 per cent. and realized £4400.

31. Two bills of exchange, one £120, payable in 6 months, and the other £150, payable in 9 months, were discounted by a Banker for £8. 10s. Required the rate per cent. he charged.

Ans. £5. 1s. $10\frac{1}{4}$d.

32. A and B travelled on the same road, at the same rate, from Leeds to London: at the fiftieth mile stone from London, A overtook a flock of geese, which was proceeding at the rate of three miles in two hours, and two hours afterwards met a stage-waggon, which was moving at the rate of nine miles in four hours: B overtook the same geese at the forty-fifth mile stone, and met the same waggon exactly forty minutes before he came to the thirty-first mile stone: where was B when A reached London?

Ans. 25 miles from London.

ARITHMETICAL AND GEOMETRICAL PROGRESSIONS.

For the Rules and Application of Arithmetical and Geometrical Progression the reader is referred to *Bonnycastle's Arithmetic*, in which he will find these subjects well digested.

1. Three days after a waggon, which travelled 9 miles a-day, was dispatched, a person started to overtake it; in order to do this he was to travel two miles the first day, three the second, and so on. In how many days will he overtake the waggon?

Let x = the number of days required;

then, by Bonnycastle's Arithmetic, problem 3rd, $(x-1)+2$ = the number of miles the man travelled the last day;

and, by problem 4th, $(x^2+3x)\div 2$ = the whole distance he travelled.

Also, $x+3$ = the number of days the waggon was on the road before the man overtook it;

then, $9x+27$ the distance the waggon had travelled;

$$\therefore \frac{x^2+3x}{2}=9x+27, \text{ or } x^2-15x=54;$$

comp. the sq. $x^2-15x+\frac{225}{4}=\frac{225}{4}+54=\frac{441}{4}$;

extracting the root, $x-\frac{15}{2}=\frac{21}{2}$; $\therefore x=18$.

2. Find three numbers in arithmetical progression such, that the square of the first, *plus* the product of the other two, shall be 16; and the square of the mean, *plus* the product of the extremes, shall be 17.

Let x = the second term,

and y = the common difference;

then, $x-y$, x, and $x+y$, will denote the three numbers;

$\therefore$ by the question, $2x^2-xy+y^2=16$,

and $2x^2-y^2=17$;

$\therefore$ by addition, $4x^2-xy=33$;

by transposition, $xy=4x^2-33$;

squaring both sides, $x^2y^2=16x^4-264x^2+1089$;

from the second equation, $y^2=2x^2-17$;

multiplying this equation by x^2, $x^2y^2=2x^4-17x^2$;

$\therefore$ $16x^4-264x^2+1089=2x^4-17x^2$;

by transposition, $14x^4-247x^2=-1089$;

completing the square,

$$x^4-\frac{247x^2}{14}+\left|\frac{247}{28}\right|^2=\frac{61009}{784}-\frac{1089}{14}=\frac{25}{784};$$

extracting the root, $x^2-\frac{247}{28}=\frac{5}{28}$;

$\therefore$ $x=3$, and $y=1$;

whence, the numbers are 2, 3, and 4.

3. A charitable person relieved four poor women, to the first of whom he gave a certain number of shillings, to the second he gave two shillings more than to the first, to the third he gave two shillings more than to the second, and to the fourth he gave two shillings more than to the third: now the continued product of their money, multiplied in shillings, amounts to £173. 5s. Required the sum given to each.

Let $x-3$, $x-1$, $x+1$, and $x+3$ denote the number of shillings respectively given to each;

$\therefore$ by the qu. $\overline{x-3}\times\overline{x-1}\times\overline{x+1}\times\overline{x+3}=3465$,

or $x^4-10x^2+9=3465$;

completing the square, $x^4-10x^2+25=3481$;

extracting the root, $x^2-5=59$; $\therefore\ x=8$;

whence, he gave 5s. to the first, 7s. to the second, 9s. to the third, and 11s. to the fourth.

4. It is required to find five numbers in arithmetical progression, whose sum is 25 and product 2520.

Let x = the third term,

and y = the common difference ;

then, $x-2y$, $x-y$, x, $x+y$, and $x+2y$ will denote the five numbers ;

$\therefore\ 5x=25$; and $x=5$.

Also, $(x^2-4y).(x^2-y^2).x=2520$;

substituting the value of x in this equation,

$(25-4y^2).(25-y^2).5=2520$;

$$\therefore\ y^4-\frac{125}{4}y^2=-\frac{121}{4}\ ;$$

comp. the sq. $y^4-\frac{125}{4}y^2+\left|\frac{125}{8}\right|^2=\frac{15625}{64}-\frac{121}{4}=\frac{13689}{64}$

extracting the root, $y^2-\frac{125}{8}=\pm\frac{117}{8}$;

$\therefore$ by taking the negative quantity, $y=1$;

whence, the numbers are 3, 4, 5, 6, and 7.

5. Divide £190 among A, B, and C, so that their respective shares may be in geometrical proportion, and that C's share may exceed A's by £50. Required each person's share.

Let x = A's share in pounds ;

then, $x+50$ = C's share,

and $190-2x-50$, or $140-2x$ = B's share ;

$\therefore\ x : 140-2x :: 140-2x : x+50$;

then $19600-560x+4x^2=x^2+50x$;

by transposition, $x^2-\frac{610x}{3}=-\frac{19600}{3}$;

completing the sq. $x^2-\frac{610}{3}x+\frac{93025}{9}=\frac{34225}{9}$;

extracting the root, $x-\frac{305}{3}=\pm\frac{185}{3}$;

taking the negative value, $x=40$;

whence, the other shares are £60 and £90.

6. Required four numbers in geometrical progression, the sum of the first and second being 20, and the sum of the third and fourth 45.

Let x = the first term, and y = the third;

then, $20-x$ = the 2nd term, and $45-y$ = the 4th.

$\therefore x : 20-x :: y : 45-y$,

or $45x-xy=20y-xy$, or $20y=45x$; $\therefore y=\frac{9x}{4}$,

and $xy=400-40x+x^2$;

substituting the value of y in the second equation,

$\frac{9x^2}{4}=400-40x+x^2$; $\therefore \frac{3x}{2}=20-x$;

and $3x=40-2x$;

$\therefore x=8$, and $y=\frac{9x}{4}=18$;

whence, the numbers are 8, 12, 18, and 27.

Or thus:

Let x = the first term, and y the common ratio,

then, x, xy, xy^2, and xy^3 will denote the four numbers;

$\therefore$ by the question, $x+xy=20$, and $xy^2+xy^3=45$,

or $x.(1+y)=20$, and $xy^2.(1+y)=45$;

$\therefore 1+y=\frac{20}{x}$, and $1+y=\frac{45}{xy^2}$;

Axiom 5. $\frac{20}{x}=\frac{45}{xy^2}$, or $20=\frac{45}{y^2}$; $\therefore y=1\frac{1}{2}$;

whence, the numbers are 8, 12, 18, and 27.

7. Divide £700 among A, B, C, and D, so as their shares may be in geometrical progression, and that the difference between A's and D's may be to the difference between B's and C's as 37 to 12. Required each person's share.

Let x = A's share in pounds, and y = the common ratio;

then, xy = B's, xy^2 = C's, and xy^3 = D's;

$\therefore$ by the question, $x+xy+xy^2+xy^3=700$.

Also, $xy^3-x : xy^2-xy :: 37 : 12$;

$\therefore 12.(xy^3-x)=37.(xy^2-xy)$,

or $12.(y^3-1)=37.(y^2-y)$;

dividing this equation by $12.(y-1)$,

$$y^2+y+1=\frac{37}{12}y \; ; \; \therefore \; 12y^2+12y+12=37y \; ;$$

by transposition, &c. $y^2-\frac{25}{12}y=-1$;

completing the square, $y^2-\frac{25}{12}y+\overline{\frac{25}{24}}\Big|^2=\frac{49}{576}$;

extracting the root, $y-\frac{25}{24}=\frac{7}{24}$; $\therefore \; y=\frac{4}{3}$;

substituting this value of y in the first eq. $x=108$;

whence, A's share £108, B's £144, C's £192, and D's £256.

8. There is a number consisting of three digits, in geometrical progression, the first of which is to the second as the second to the third, the number itself is to the sum of its digits as 124 to 7, and if 594 be added to it, the digits will be inverted. Required the number.

Let x = the first digit, or that in the hundredth's place, and y = the ratio;

then x, xy, and xy^2 will denote the three digits;

$\therefore$ by the question,

$100x+10xy+xy^2 : x+xy+xy^2 :: 124 : 7$;

dividing the first and second terms by x,

$100+10y+y^2 : 1+y+y^2 :: 124 : 7$;

Theorem 4, $99+9y : 1+y+y^2 :: 117 : 7$;

Theorem 10, $11+y : 1+y+y^2 :: 13 : 7$;

$\therefore\ 13y^2+13y+13=77+7y$;

by transposition, $13y^2+6y=64$;

completing the sq. &c. y is found to be equal to 2.

Also, by the question,

$100x+10xy+xy^2+594=100xy^2+10xy+x$;

by transposition, $99x+594=99xy^2$;

dividing by 99, $x+6=xy^2$;

substituting the value of y in this equation,

$x+6=4x$; $\therefore\ x=2$;

hence, the number is 248.

EXAMPLES FOR PRACTICE

In Arithmetical and Geometrical Progression.

1. The three sides of a triangle, whose perimeter is 15, are in arithmetical progression, and the sum of the first and second terms, is to the sum of the second and third, as 2 to 3. Required the sides.

Ans. 3, 5, and 7.

2. The three sides of a triangle, whose perimeter is 24, are in arithmetical progression, and the product of the first and second, is to the product of the second and third, as 3 to 5. Required the sides of the triangle.

Ans. 6, 8, and 10.

3. A person had seven sons, who had each saved a certain number of shillings, which were in arithmetical progression: the number saved by the second was 8, and the number saved by the oldest was 18. Required the savings of each.

Ans. 6, 8, 10, 12, 14, 16, and 18.

4. There is a number consisting of three digits, which are in arithmetical progression, which being divided by the sum of its digits, the quotient is $28\frac{3}{4}$; but if 198 be added to it, the digits will be inverted. Required the number.

Ans. 345.

5. The sum of four numbers, in arithmetical progression, is 56, and the sum of their squares = 864. Required the numbers.

Ans. 8, 12, 16, and 20.

6. Find four numbers in arithmetical progression, the sum of the squares of the means being 400, and the sum of the squares of the extremes 464.

Ans. 8, 12, 16, and 20.

7. After A, who travelled four miles an hour, had been on his journey $2\frac{1}{4}$ hours, B set out to overtake him by going $4\frac{1}{2}$ miles the first hour, $4\frac{3}{4}$ the second, 5 the third, &c. How many hours was B in overtaking A?

Ans. 8.

8. The sum of the first and second of three numbers in geometrical progression is 12, and the sum of the second and third is 36. Required the numbers.

Ans. 3, 9, and 27.

9. It is required to find three numbers in geometrical progression, whose sum is 7, and the sum of their squares 21.

Ans. 4, 2, and 1.

10. Required three numbers in geometrical progression, whose product is 27, and the sum of their squares 91.

Ans. 1, 3, and 9.

11. A person has four sons, whose ages are in arithmetical progression, the product of the youngest son's age multiplied by that of the oldest is 40, and the quotient of the third son's age divided by the second is $1\frac{1}{2}$. Required each son's age.

Ans. 4, 6, 8, and 10 years.

12. A stationer sold seven books, their prices in shillings being in decreasing arithmetical progression; the price of the second was 13s., and the number of shillings that the first cost, was to the number the seventh cost, as 5 to 1. Required the price of each.

Ans. 15s., 13s., 11s., 9s., 7s., 5s., and 3s.

13. Required four numbers in geometrical progression, the sum of whose means is 30, and the sum of the extremes 45.

Ans. 5, 10, 20, and 40.

14. A person bought four reams of paper, the particular prices of which (in shillings) were in geometrical proportion: the number of shillings the first cost, multiplied by the number the second cost, makes £1. 12s., and the product of the third and fourth, multiplied in shillings, is £25. 12s. Required the price of each.

Ans. The first 4s., second 8s., third 16s., and fourth 32s.

15. It is required to distribute £4. 10s. among four persons, A, B, C, and D, so that their respective shares, in shillings, may be in geometrical progression; the sum received by A and C was 30s. Required the sum received by each.

Ans. A received 6s., B 12s., C 24s., and D 48s.

16. Required four numbers in geometrical progression such, that if the first and second be multiplied by 2, and 6 added to the third, they will form a series of numbers in arithmetical progression.

Ans. 3, 6, 12, and 24.

17. A person bought four remnants of cloth, whose respective lengths in yards were in arithmetical progression, and found that if the first had contained 2 yards, the second 3 yards, the third 9 yards, and the fourth 25 yards more than they did, their respective lengths would have been in geometrical progression. How many yards did he buy?

Ans. 36 yards.

18. What five numbers are in geometrical progression, the sum of the first and third being 15, and the sum of the third and fifth 60?

Ans. 3, 6, 12, 24, and 48.

19. Two travellers, A and B, set out together from the same place; A goes eight miles the first day, twelve the second, &c., B goes one mile the first day, four the second, nine the third, &c. according to the square of the number of days he travelled. In what time will B overtake A?

Ans. 7 days.

20. Two persons, A and B, at the distance of 462 miles, set out at the same time to meet each other: A goes one mile the first day, two the second, three the third, &c., and B travels each day a number of miles which is equal to the cube of the number of miles that A travels on the same day. In what time will they meet?

Ans. 6 days.

SOLUTION OF CUBIC AND OTHER EQUATIONS BY APPROXIMATION.

DEFINITIONS.

1. A *Cubic Equation* is one that contains the third power of the unknown quantity, as $x^3+ax^2+bx=c$.

2. A *Biquadratic Equation* is one that contains the fourth power of the unknown quantity, as $x^4+ax^3+bx^2+cx=d$.

3. An Equation of the *fifth* power, is one that contains the fifth power of the unknown quantity, &c.

Many ingenious rules, though some of them very tedious, have been given for the solution of cubic and other equations of higher powers, but they may be more readily solved by approximation, sometimes called "trial and error."

RULE.

Substitute for x in the given equation, two numbers as near its true value as possible, observing that one of them be greater and the other less than its true root or value, and mark the separate results; then, as the difference between these results is to the difference between the numbers substituted, so is the difference between the true result and either of the former to the respective correction of each: this correction being added to the number when too small, or subtracted from it when too large, will give the root or value of x nearly.

The number thus found, with any other that may be supposed to approach still nearer to the true root or value of x, may be assumed for another operation, which may be repeated till the root shall be determined to any degree of exactness that may be required.

EXAMPLES.

1. Given $x^3+x^2+x=100$, to find the value of x.

Here, by a few trials, x is found to be more than 4, and less than 5; therefore, if these two numbers be substituted in the given equation, and the results calculated as follows:

By the first supposition		By the second supposition	
$x=$	4	$x=$	5
$x^2=$	16	$x^2=$	25
$x^3=$	64	$x^3=$	125
	84	results	155

155	5	100 true result,
84	4	84

as 71 : 1 :: 16 : .2253+

therefore, $x=4+.2253=4.2253$, which approximates nearly to the true value.

If now 4.2 and 4.3 be taken as the assumed numbers, and the operation repeated,

By the first supposition		By the 2nd. supposition	
$x=$	4.2	$x=$	4.3
$x^2=$	17.64	$x^2=$	18.49
$x^3=$	74.088	$x^3=$	79.507
	95.928	results	102.297

102.297	4.3	102.297	
95.228	4.2	100	true result,

6.369 : .1 :: 2.297 : .036;

therefore, $x=4.3-.036=4.264$, nearly.

Again, if still greater accuracy be required, let 4.264 and 4.265 be the assumed numbers, then

$x = 4.264$		$x = 4.265$	
$x^2 = 18.181696$		$x^2 = 18.190225$	
$x^3 = 77.526752$		$x^3 = 77.581310$	
99.972448	results	100.036535	
100.036535	4.265	100	true result,
99.972448	4.264	99.972448	
.064087	: .001 ::	.027552	: .0004299;

as .064087 : .001 :: .027552 : .0004299; consequently, $x = 4.264 + .0004299 = 4.2644299$, very nearly the truth.

EXAMPLES FOR PRACTICE.

1. Given $x^3 - 15x^2 + 63x = 50$, to find the value of x.

Ans. $x = 1.028039$.

2. Given $x^3 - 12x = 84$, to find the value of x.

Ans. $x = 5.2822$.

3. Given $x^3 - 17x^2 + 54x - 350 = 0$, to find the value of x.

Ans. $x = 14.954$.

4. Given $x^6 - 3x^4 - 2x^2 = 8$, to find the value of x.

Ans. $x = 2$.

5. Given $2x^4 - 16x^3 + 40x^2 - 30x + 1 = 0$, to find the value of x.

Ans. $x = 1.284724$.

6. Given $x^5 + 2x^4 + 3x^3 - 4x^2 + 5x - 1 = 54320$, to find the value of x.

Ans. $x = 8.4144$.

7. Find two numbers, whose product multiplied by the greater will make 405, and their difference multiplied by the less will make 20.

Ans. 9 and 5.

INDETERMINATE ANALYSIS,

OR UNLIMITED PROBLEMS.

It has already been determined how the value of one unknown quantity may be found from one equation, and also how the values of two unknown quantities may be found from two equations, &c. having as many independent equations as there are unknown quantities to determine, which must be the case when the given equations or questions require some certain or definite answer.

When a question does not furnish as many equations as there are unknown quantities to be determined, some of these must remain undetermined, and depend on the will; for which reason, such questions are said to be indeterminate, and frequently admit of many answers, forming the subject of a particular and useful branch of Algebra, called *Indeterminate Analysis.*

CASE I.

When the given equation contains two unknown quantities.

RULE.

1. *Find the value of one of the unknown quantities in terms of the rest, as in step first, in the first example.*

2. *Divide the numerator by the denominator, if divisible, as in step second, and put the remainder equal to a whole number, w, if the coefficient of the unknown quantity in this remainder be* 1, *as in the fourth step.*

3. *Clear this equation of the fraction, and find the value of the unknown quantity, in terms of the rest, as in the fifth step.*

4. *Then w may be taken any number that will answer the conditions of the question, as in step sixth.*

Or the value of this unknown quantity may be substituted in the first step of the equation, as in the seventh step, where w may be taken any number that will answer the conditions of the question, as in the eighth step.

5. *But if the coefficient of the unknown quantity be more than one after the division, or that the numerator cannot be divided by the denominator, as mentioned in the second part of this rule (as in the fourth step of the second and third examples), take any multiple of this quantity that may be convenient, and add it to the unknown quantity, if the unknown quantity in the fractional part be negative, but subtract it if the unknown quantity be positive, in order to exterminate the coefficient of the unknown quantity in the fractional part, then put this sum or remainder equal to w, and proceed by the third part of the rule.*

EXAMPLES.

1. Given $4x-5y=10$, to find the values of x and y in whole numbers.

1. By transposition, $x=\frac{10+5y}{4}$, which must be a whole number.

2. Dividing the numerator by the denominator, $\frac{10+5y}{4}=2+y+\frac{2+y}{4}$, which must also be a whole number.

3. But $2+y$ being a whole number, it may be rejected, and the remainder $\frac{2+y}{4}$ (the unknown

quantity y having 1 for a coefficient,) must be put equal to a whole number w.

4. $\therefore$ Let $\frac{2+y}{4}=w$.

5. Clearing this equation of the fraction, &c. $y=4w-2$.

6. Let w be taken any number that will answer the conditions of the question; $\therefore$ if $w=1$, then $y=2$ and $x=5$, two numbers which will answer the conditions of the question.

7. Or by substituting the value of y in the equation of the first step of this solution,
$$x=\frac{10+20w-10}{4};$$

8. Then, as above, if $w=1$, $x=5$, and $y=2$, two numbers that will answer the conditions of the question.

N. B. Several sets of numbers may be found that will answer the conditions of the question, by assuming w any other value, as follows:

If $w=2$, then $x=10$, and $y=6$.

$w=3$, then $x=15$, and $y=10$.

$w=4$, then $x=20$, and $y=14$, &c. all of which numbers will answer the conditions of the question.

2. Given $3x=8y-16$, to find the values of x and y in whole numbers.

1. Here $x=\frac{8y-16}{3}$, which must be a whole number.

2. Dividing the numerator by the denominator, $\frac{8y-16}{3}=2y-5+\frac{2y-1}{3}$, which must be a whole number.

4. Then $2y-5$ being a whole number, it remains to make the remainder, $\frac{2y-1}{3}$, a whole number: but the unknown quantity in this remainder has 2 for a coefficient, therefore, if this remainder be subtracted from y, or its equal, $\frac{3y}{3}$, as follows, $\frac{3y}{3}-\frac{2y-1}{3}=\frac{y+1}{3}$, which must be a whole number; here the coefficient of the unknown quantity y being 1, the expression may be put equal to a whole number w.

5. Let $\frac{y+1}{3}=w$;

6. Therefore $y=3w-1$.

7. Let w be taken equal to 2, then $y=5$, and $x=8$, two numbers that will answer the conditions of the question.

8. Or, by substituting the value of y in the first step of this solution, as in the first example, $x=\frac{24w-8-16}{3}$;

9. Then, as above,

If w be taken 2, then $y=5$, and $x=8$,
w 3, $y=8$, $x=16$,
w 4, $y=11$, $x=24$,
w 5, $y=14$, $x=32$, &c.

numbers that will answer the conditions of the question.

3. Given $17x+15y=130$, to find x and y in whole positive numbers.

1. Here $y=\frac{130-17x}{15}$, which must be a whole number.

2. Dividing the numerator by the denominator, $\frac{130-17x}{15}=8-x+\frac{10-2x}{15}$, which must be a whole number.

3. But $8-x$ being a whole number, it may be rejected, and the remainder, $\frac{10-2x}{15}$, must be made equal to a whole number w.

4. By the fifth part of the rule, the unknown quantity, x, in this remainder, having more than 1 for a coefficient, therefore, multiply $\frac{10-2x}{15}$ by 7, and the product will be $\frac{70-14x}{15}$, which being added to x, or which is the same thing, to $\frac{15x}{15}$, as follows, $\frac{15x}{15}+\frac{70-14x}{15}=\frac{70+x}{15}$.

5. Let $\frac{70+x}{15}=w$;

6. Then $70+x=15w$; $\therefore\ x=15w-70$;

7. Let w be taken $=5$;

8. Then $x=5$, and $y=3$, two numbers that will answer the conditions of the question.

EXAMPLES FOR PRACTICE.

1. Given $3x+2y=26$, to find the values of x and y in whole positive numbers.

Ans. $x=6$, and $y=4$.

2. Given $4x-5y=10$, to find the values of x and y in whole numbers.

Ans. $x=5$, and $y=2$.

3. Given $12x=138-7y$, to find the values of x and y in whole numbers.

Ans. $x=8$, and $y=6$.

4. Given $87x+256y-1908=14102$, to find the least value of x, and the greatest of y, in whole numbers.

Ans. $x=30$, and $y=50$.

5. Divide 100 into two such parts, that the one may be divisible by 7, and the other by 11.

Ans. 56 and 44.

6. Two boys having together 100 marbles; one says to the other, 'When I count my marbles by eights, there is an overplus of 7:' the other replies, 'If I count my marbles by tens, I find the same overplus of 7.' How many marbles had each?

Ans. The first boy had 23, and the second 77.

7. A draper bought two sorts of cloth, consisting of blue and black, for £88. 10s.: for the blue he paid 31s. per yard, and for the black 21s. per yard. How many yards did he buy of each?

Ans. 9 yards of blue, and 71 of black,
or 30 do. 40 do.

8. In what year of our Lord was the cycle of the sun 23, the cycle of the moon 14, and the cycle of the Roman Indiction 4?

Ans. 1666.

9. How much wheat at 5s. 4d. per bushel, and rye at 3s. 7d. per bushel, will compose a mixture worth 4s. 7d. per bushel?

Ans. 4 bushels of wheat, and 3 of rye,
or 8 do. 6 do.
or 12 do. 9 do.

N. B. Any two numbers in the proportion of 4 to 3 will answer the question.

CASE II.

When the given equation contains three or more unknown quantities.

RULE.

Find the limit of that quantity which has the greatest coefficient, and then find the different values by Case I. as in the following example.

Given $3x+4y+5z=62$, to find the values of x, y, and z, in whole numbers.

Here, in order to find the limit or number of values that z may have (z having the greatest coefficient), take the values of x and y, each equal to 1; then $z=\frac{62-3-4}{5}=\frac{55}{5}=11$, the limit or number of values that z can have, though they may be considerably less.

Then, by Case I,

$$x=\frac{62-4y-5z}{3}=20-y-z+\frac{2-y-2z}{3};$$

by the fifth part of the first rule, the remainder, $\frac{2-y-2z}{3}$, must be a whole number, but z having more than 1 for a coefficient, and the $2z$ being a negative quantity, this remainder must be added to z in order to exterminate the coefficient; thus, $\frac{3z}{3}+\frac{2-y-2z}{3}=\frac{z+2-y}{3}$, which must be a whole number;

let $\frac{z+2-y}{3}=w$; then $z+2-y=3w$;

$\therefore y=z+2-3w$; and by taking $w=0$, $y=z+2$.

Now, by the limit, z may be taken any number from 1 to 11, that will answer the conditions of the question;

if $z=1$, y will $=3$, and $x=16$,
if $z=2$, y will $=4$, and $x=15$,
if $z=3$, y will $=5$, and $x=14$,
if $z=4$, y will $=6$, and $x=13$, &c.

EXAMPLES FOR PRACTICE.

1. Given $3x+5y-12z=3$, to find the values of x, y, and z, in whole numbers.

Ans. $x=4$, $y=3$, and $z=2$.

2. Given $5x+2y+3z=22$, to find the values of x, y, and z, in whole numbers.

Ans. $x=3$, $y=4$, and $z=5$.

3. Given $4x-6y+3z=2$, to find the values of x, y, and z, in whole numbers.

Ans. $x=2$, $y=4$, and $z=5$.

4. A silver-smith has three kinds of silver, the first of 7 oz., the second of $5\frac{1}{2}$ oz., and the third of $4\frac{1}{2}$ oz. fine, per mark of 8 oz. How many ounces of each must he take to form a mixture to weigh 30 marks, at 6 oz.?

Ans. 12, 15, and 3; or 14, 10, and 6; or 16, 5, and 9.

5. Required three numbers such, that if the first be multiplied by 3, the second by 5, and the third by 7, the sum of their products will be 560; but if the first be multiplied by 9, the second by 25, and the third by 49, the sum of their products will be 2920.

Ans. 15, 28, and 15; or 50, 10, and 30.

6. A person bought oxen, cows, calves, and sheep, to the amount of 100 in number, for £[illegible]9: for each ox he gave £10, for each cow £5, for each calf £2, and for each sheep 10s. How many did he buy of each?

	or	or	or	or	or	or	or	or	or	
Oxen,	1,	1,	1,	1,	1,	1,	1,	1,	4,	4.
Ans. Cows,	1,	2,	3,	4,	5,	6,	7,	8,	1,	2.
Calves,	24,	21,	18,	15,	12,	9,	6,	3,	5,	2.
Sheep,	74,	76,	78,	80,	82,	84,	86,	88,	90,	92.

7. How many different ways may wine at 6d. 8d. 10d. and 16d. per quart, be mixed so as to make 100 quarts worth 12d. per quart?

Ans. 312.

THE APPLICATION OF ALGEBRA

TO THE SOLUTION OF

GEOMETRICAL PROBLEMS.

In order to solve Geometrical Problems Algebraically, draw a figure to represent the conditions of the problem, and consider it the true one.

Then, let the unknown line or lines, which appear the easiest to find, be represented by the common algebraical symbols; connect these, by the different rules and theorems in Euclid's Elements, with the known quantities or lines, and form as many independent equations as there are unknown quantities, which may be solved by the rules already given.

In some cases, after the figure is drawn, it requires to be prepared for solution, by producing and drawing such lines as may be wanted, in order to obtain a solution; but as no general rule can be given for preparing the figures, and making the proper substitutions, so as to bring out the most simple conclusions, it may not be improper, in the learner to attempt the solution of the same problem different ways, and apply that method which succeeds the best to similar cases.

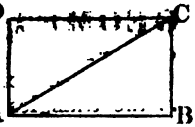
Before the Student attempts the solution of Geometrical Problems by Algebra, he must be tolerably versed with the Elements of Euclid.

The Author gives preference to Bonnycastle's Euclid.

PROBLEM I.

Given the base of a right angled triangle 4 chains and the sum of the hypothenuse and perpendicular 8 chains. Required the area of the triangle.

In the right angled triangle D A B C, let x = the perpendicular B C; then $8-x$ = the hypothenuse A C;



(Eu. 47, 1.) $AB^2+BC^2=AC^2$,

or $16+x^2=64-16x+x^2$;

by transposition, $16x=48$; $\therefore x=3$;

whence, the area of the triangle is 6 square chains, or 2R. 16P.

PROBLEM II.

The base of a right angled triangle is 4 chains, and the difference between the hypothenuse and perpendicular 2 chains. Required the hypothenuse and perpendicular.

In the above figure, let x = the perpendicular B C; then $x+2$ = the hypothenuse A C;

(Eu. 47, 1.) $AC^2=AB^2+BC^2$, or $x^2+4x+4=16+x^2$;

by transposition, $4x=12$; $\therefore x=3$;

whence, the hypothenuse is 5, and perpendicular 3.

PROBLEM III.

The diagonal of a rectangle is 50, and the sum of the four sides 140. Required its area.

In the above figure, let x = AB, the length; then $70-x$ = BC, the breadth;

$\therefore$ (Eu. 47, 1.) $AB^2+BC^2=AC^2$,

or $x^2+4900-140x+x^2=2500$;

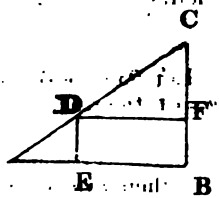
by transposition, $2x^2-140x=-2400$, this equation solved gives $x=40$;

whence, the area is 1200.

The Author gives previous

PROBLEM IV.

The area of a right angled triangle is 600, and the length and breadth of the inscribed rectangle 24 and 8. Required the sides of the triangle.



In the right angled triangled ABC, let $x=$ the perpendicular BC; then $x-8=$CF;

$\therefore$ the triangles ABC and DFC being similar, by Euc. VI. 4, CF : DF :: BC : AB,

or $x-8 : 24 :: x : \frac{24x}{x-8}=$AB;

then, $\frac{24x}{x-8}\times\frac{x}{2}=600$, or $x^2-50x=-400$, which equation being solved, gives $x=40$.

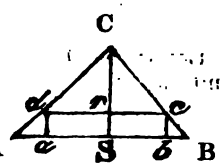
Also, by similar triangles, FC : DF :: BC : AB,

or 32 : 24 :: 40 : 30;

whence, the three sides are 30, 40, and 50.

PROBLEM V.

The base of a plane triangle is 40, its area 400, and the area of the inscribed rectangle 150. Required the sides of the rectangle.



Let ABC be the triangle, and *abcd* the inscribed rectangle.

Let $x=ab$ or cd; then by similar triangles, (Eu. VI. 4.) AB : cd :: SC : Cr;

or $40 : x :: 20 : \frac{x}{2}=Cr$.

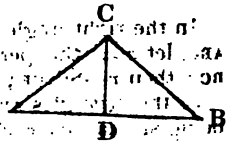
then $x.(20-\frac{x}{2})=150$; $\therefore x^2-40x=-300$;

this equation being solved, $x=30$;

whence its breadth is 5 and length 30.

PROBLEM VI.

The area of an isosceles triangle, is 675, and the sum of its equal sides is equal to the sum of the base and perpendicular. Required the sides and perpendicular of the triangle.



Let $2x$ = AB the base, and y = CD the perpendicular;

then $xy = 675$; and $\sqrt{x^2 + y^2}$ = AC or BC, one of the equal sides; $\therefore 2\sqrt{x^2+y^2} = 2x + y$;

squaring each side, $4x^2 + 4y^2 = 4x^2 + 4xy + y^2$

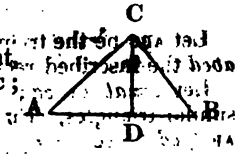
by transposition, &c. $3y = 4x$.

These equations being solved, x is found to be 22.5 and $y = 30$;

Whence the base is 45, the perpendicular 30, and each of the equal sides 37.5.

PROBLEM VII.

Given the segments of the base, made by the perpendicular from the vertex angle upon the base of a plane triangle, 20 and 15; and the ratio of the sides as 5 to 3. Required the sides.



Let AD be the greater segment, and BD the less; put x = AC;

then $5 : 4 :: x : \frac{4x}{5}$ = BC.

$\therefore$ (Eu. 47. 1) $AC^2 - AD^2 = DC^2 = BC^2 - BD^2$,

or $x^2 - 400 = \frac{16x^2}{25} - 225$;

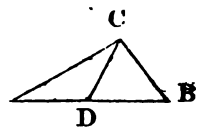
this equation being solved, x is found = 22.0479276.

Whence AC = 22.0479276 and BC = 17.63834208.

PROBLEM VIII.

Given the base 20, the sum of the sides 26, and the line drawn from the verticle angle to the middle of the base 8.5 to determine the triangle.

Let AB represent the base, AC and BC the two sides, and CD the line drawn from the vertical angle to the middle of the base. Put x=AC; then 26—x=BC.



Then, by Bonnycastle's Euclid or Geometry, Book 2. Prop. 13.

$$AC^2+BC^2=2CD^2+2AD^2,$$

or $x^2+676-52x+x^2=144.5+200$;

by transposition, &c. $x^2-26x=-165.75$;

this equation solved, $x=14.80277564$. and BC=11.19722436.

PROBLEM IX.

Given the sides and the line bisecting the vertical angle, of any plane triangle, to find the base.

N. B. The above figure being similar to the one required for this question, the Author has considered it the real figure in the following solution.

Put the side AC=a; BC=b; CD, (the line which bisects the vertical angle ACB) =c, and the base AB=x; then, by (Euc. 3. VI.)

$$AC : CB :: AD : DB,$$

or $a : b :: AD : DB$.

And by Proportional Theorem 3. page 31.

$$a+b : b :: x : DB = \frac{bx}{a+b}$$

Also, $CB : AC :: BD : AD$,

or $b : a :: BD : AD$

$\therefore$ (Theorem 3.) $a+b : a :: x : AD = \frac{ax}{a+b}$.

Also, by (Euc. 13. VI.) AB × BB + DC² = AC × BC,

$$\text{or } \left(\frac{ax}{a+b}\right)\times\left(\frac{bx}{a+b}\right)+c^2=ab;$$

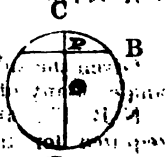
$$\therefore \frac{abx^2}{(a+b)^2}=ab-c^2;$$

then, $abx^2=(a+b)^2\times(ab-c^2)$;

$$\therefore x=(a+b).\sqrt{\frac{ab-c^2}{ab}}.$$

PROBLEM X.

A Carpenter's Apprentice applied to his Master to know the radius he must take to strike the arc of a centre for the top of a circular door, its width being forty-eight inches and the arch to rise fifteen inches. The master not being able to inform him, he begs that the young Algebraist will give a general solution to the above, so as he may in future be enabled to find the radius of any circular centre for an arch.



In the annexed figure, let AB denote the width of the door, PC the height of the arch or centre, ACB the arch or centre, and DC the diameter of the circle, of which the arch or circular door top ACB is a part.

Put AB = 2a, PC = b and BC = x.

Then PD = x + b; ∴ by the property of the circle, Euc. 35. III.

$$\text{CP} \times \text{PD} = \text{AP} \times \text{PB}, \text{ or } b.(x+b)=a^2;$$

$$\therefore bx-b^2=a^2; \text{ or } bx=a^2+b^2;$$

$$x=\frac{a^2+b^2}{b}.$$

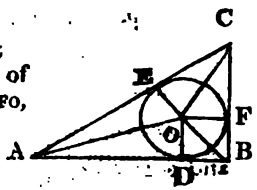
Then substituting the values of a and b, in the equation $x=\frac{24^2+15^2}{15}=\frac{576+225}{15}=53.4$ the diameter of the circle, half of which is 26.7 = BC, the radius required.

NOTE.—This question is similar to question 42, page 74.

PROBLEM XI.

Given the perimeter of a right angled triangle, and the radius of the inscribed circle; to find the sides of the triangle.

Let ABC denote the right angled triangle, o the centre of the inscribed circle, DO, EO, or FO, the radius.



Prepare the figure, for solution, by drawing the lines AO, BO, and CO; then AE=AD, CE=CF, and BD=BF.

Then put the given perimeter=p, the radius of the inscribed circle=r, AE or its equal AD=x, and CE or CF=y.

By the Problem,

$$AD+DO=AB, \text{ or } x+r=AB;$$
$$AE+EC=AC, \text{ or } x+y=AC;$$
$$\text{and } CF+FO=BC, \text{ or } y+r=BC.$$
$$\therefore x+r+x+y+y+r=p, \text{ or } 2x+2y+2r=p;$$
$$\text{by transposition, \&c. } x+y=\frac{1}{2}p-r;$$
$$x=\frac{1}{2}p-r-y.$$

By Euc. 47. 1. $(x+r)^2+(y+r)^2=(x+y)^2$,
or $x^2+2rx+r^2+y^2+2yr+r^2=x^2+2xy+y^2$,
by transposition and division,
$$r.(x+y)=xy-r^2.$$

Substituting the value of x in this equation,

$$y^2-\left(\frac{1}{2}p-r\right).y=-\frac{1}{2}pr;$$

completing the square,

$$y^2-(\tfrac{1}{2}p-r)\,y+\tfrac{1}{4}(\tfrac{1}{2}p-r)^2=\tfrac{1}{4}(\tfrac{1}{2}p-r)^2-\tfrac{1}{2}pr;$$

extracting the square root,

$$y-\tfrac{1}{2}(\tfrac{1}{2}p-r)=\mp\sqrt{\tfrac{1}{4}(\tfrac{1}{2}p-r)^2-\tfrac{1}{2}pr};$$

$$\therefore\ y=\mp\sqrt{\tfrac{1}{4}(\tfrac{1}{2}p-r)^2-\tfrac{1}{2}pr}+\tfrac{1}{2}(\tfrac{1}{2}p-r),$$

and by a similar operation,

$$x=\pm\sqrt{\tfrac{1}{4}(\tfrac{1}{2}p-r)^2-\tfrac{1}{2}pr}+\tfrac{1}{2}(\tfrac{1}{2}p-r).$$

$$\therefore \text{AB}=\pm\sqrt{\tfrac{1}{4}(\tfrac{1}{2}p-r)^2-\tfrac{1}{2}pr}+\tfrac{1}{2}(\tfrac{1}{2}p-r)+r.$$

and $\text{BC}=\mp\sqrt{\tfrac{1}{4}(\tfrac{1}{2}p-r)^2-\tfrac{1}{2}pr}+\tfrac{1}{2}(\tfrac{1}{2}p-r)+r.$

Or thus:

If the perimeter of the triangle be 120, and the radius of the inscribed circle 10.

then $x+10+y+10+x+y=120$;

$$\therefore x+y=50;\ \text{and}\ x=50-y.$$

Also, $(x+10)^2+(y+10)^2=(x+y)^2$,

or $x^2+20x+100+y^2+20y+100=x^2+2xy+y^2$;

$$\therefore\ 20x+20y=2xy-200.$$

Substituting the value of x in this equation,

$$1000-20y+20y=100y-2y^2-200;$$

$$\therefore\ y^2-50y=-600;$$

completing the square $y^2-50y+625=625-600=25$

extracting the root $y-25\mp=5$;

$$\therefore\ y=20,\ \text{and}\ x=30;$$

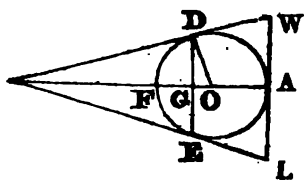
Whence AB=40, BC=30 and AC=50.

PROBLEM XII.

If a globe of two inches diameter be suspended against an upright wall, at what distance from the wall must a lighted candle be placed, so that the

enlightened part of the surface of the globe, may be equal to the area of the shadow on the wall; the light and centre of the globe being in the same horizontal line?

Let AEFD represent the globe, WAL the wall, and C the candle; then put a=AO or DO or FO, the radius, p = .7854 and CA=x, ∴ CO=$x-a$; and by



similar triangles CO : OD :: OD : OG=$\frac{a^2}{x-a}$, hence FG, the height of the enlightened segment, =OF—OG=$\frac{ax-2a^2}{x-a}$; consequently the area of the enlightened part of the globe EFD=$8p.\left(\frac{a^2x-2a^3}{x-a}.\right)$

Again CD=$\sqrt{x^2-2ax}$; then by similar triangles CD : DO :: AC : AW=$\frac{ax}{\sqrt{x^2-2ax}}$= the radius of the shadow on the wall, and $4p\times\frac{a^2x}{x-2a}$=its area; ∴ by the question, $8p\times\frac{a^2x-2a^3}{x-a}=4p\times\frac{a^2x}{x-2a}$, which by reduction becomes $x^2-7ax=-8a^2$; whence x= 5.56155.

PROBLEMS FOR PRACTICE.

1. The base of a right angled triangle is 36, and the difference between the perpendicular and hypothenuse 18. Required the area of the triangle.

Ans. 486.

2. The difference between the base and hypothenuse of a right angled triangle is 9, and the difference

between the hypothenuse and perpendicular 18. Required the sides of the triangle.

Ans. Base 36, perpendicular 27, and hypothenuse 45.

3. The difference between the base and perpendicular of a right angled triangle is 9, and the perpendicular let fall from the right angle upon the hypothenuse 21.6. Required the sides of the triangle.

Ans. 27. 36, and 45.

4. The base of a right angled triangle is 40, and the alternate segment of the hypothenuse, made by a perpendicular from the right angle upon the hypothenuse, 18. Required the sides of the triangle.

Ans. 30, 40, and 50.

5. The perimeter of a right angled triangle is 144, and its area 864. Required the sides

Ans. 36, 48, and 60.

6. The hypothenuse of a right angled triangle is 50, and the perpendicular let fall from the right angle upon the hypothenuse 24. Required the segments made by this perpendicular.

Ans. 18, and 32.

7. In a right angled triangle the sum of the greater segment, made by a perpendicular let fall from the right angle upon the hypothenuse, and the adjacent side is 72. Required the sides of the triangle.

Ans. 30, 40, and 50.

8. The base of a right angled triangle is 40, and the perpendicular 30. Required the length and breadth of that inscribed rectangle which takes up one-half of the area of the triangle.

Ans. 20, and 15.

9. The base of a plain triangle is 30, and the perpendicular 20. Required the area of the inscribed square.

Ans. 144.

10. The diameter of a semicircle is 20, and the breadth of the inscribed rectangle 6. Required the area of the rectangle.

Ans. 96.

11. The diameter of a semicircle is 10; required the length and breadth of that inscribed rectangle, whose area is equal to one third of the area of the semicircle.

Ans. 9.6227 and 1.36035.

12. The diameter of a semicircle is 25, it is required to find a point in its arc, such, that if lines be drawn from the extremities of the diameter to that point, the triangle so formed shall be equal to one-half of the semicircle.

Ans. 11.2901 from the extremity of the diameter.

13. In an isosceles triangle two circles, whose diameters are 8 and 12, touching each other and the sides of the triangle. Required the two equal sides.

Ans. 36.7423 each of the equal sides.

14. Six equal circles, each two inches in diameter, are inscribed in an equilateral triangle, touching each other and the sides of the triangle. Required each side of the triangle.

Ans. 7.4641016.

15. The three lines drawn from the angles of a plain triangle to the middle of the opposite sides are 18, 24, and 30. Required the sides of the triangle.

Ans. 34.176; 28.844; and 20.

16. The difference of the sides of a plain triangle is 405, the difference of the segments of the base, 495, and the difference between the base and greater side 165. Required the sides of the triangle.

Ans. Base 945, greater side 780, and less side 375.

17. The difference of the sides of a plain triangle is 405, the difference of the segments of the base 495,

and the perpendicular let fall from the vertical angle 300. Required the sides.

Ans. Base 945, greater side 780, and less side 375.

18. The sum of two sides of a plain triangle is 1155, the difference of the segments of the base 495, and the perpendicular let fall from the vertical angle 300. Required the sides.

Ans. Base 945, and the sides 780, and 375.

19. The area of a plain triangle is 141750, the difference of the sides 405, and the difference of the segments of the base 495. Required the sides.

Ans. 945, 780, and 375.

20. What are the dimensions of a right cone of equal base and altitude, when its solidity and superficies are the same.

Ans. 9.70824 the diameter and altitude.

21. A triangular piece of ground, containing 12150 square yards, has a spring of water at the right angle, which runs perpendicularly to the opposite side, and crosses it at the distance of 81 yards from the nearest acute angle. If this piece of ground be divided by a line parallel to the shortest side, so that the trapezium may be to the other part as 5 to 4; at what distance from the spring will the division line cross the stream?

Ans. 100 yards.

22. Given the three chords of three arches completing a semicircle 60, 70, and 80, to find the area of the semicircle.

Ans. 7731.918.

23. There are two stone pillars, whose perpendicular heights are 20 and 30 feet, and the distance between them 60 feet. A ladder is to be placed at a certain point in the line of distance, of such a length, that from thence it may just reach the top of both the pillars. Where is that point, and what is the length of the ladder?

Ans. $34\frac{1}{5}$ feet from the less pillar, and the ladder is 39.5899 feet.

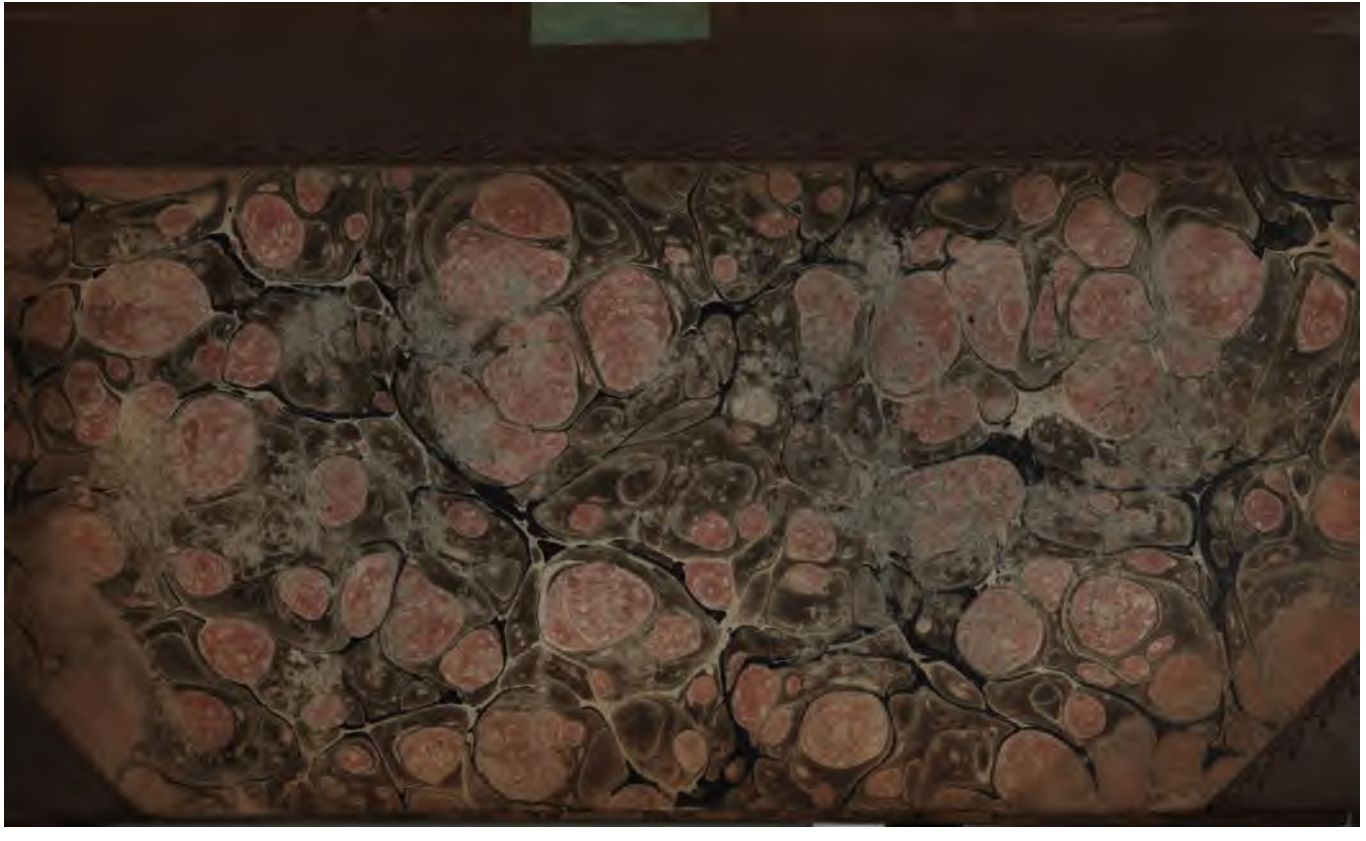
24. The perimeter of a plain triangle is 400, and its angles as the numbers 1, 2, and 3. Required the length of each side.

Ans. 177.96+; 142.79+; and 79.247+.

25. Two persons observing an object, 40 feet in height, placed on the top of York Minster, the one stood $65\frac{1}{5}$ yards, and the other $86\frac{1}{5}$ yards from it, when the object appeared to them both of the same magnitude. Required the height of the minster, supposing the height of each person's eye to be $5\frac{1}{2}$ feet.

Ans. 213.0037 feet.

26. A gentleman ordered his goldsmith to make him a hemispherical bowl, one fortieth of an inch thick throughout, and of such other dimensions, as to require seven times its weight put into it, to immerse it in common water just to the brim; but when made, it was found on trial to require one dram avoirdupois more than the weight proposed. Now, if the specific gravity of the metal was ten times that of water, what was the diameter of the bowl?

Ans. 11.9027602 inches.

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